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*A THEORY CONCERNING THE MECHANISM OF FATTY ACID OXIDATION AND SYNTHESIS, AND OF CARBON DIOXIDE FIXATION**

By Henry A. Lardy

Department of Biochemistry and Institute for Enzyme Research, University of Wisconsin, Madison

Communicated by C. A. Elvehjem, October 1, 1952

The discoveries of Lynen, *et al.*,[1] and of Racker[2] have provided a mechanistic basis for the participation of thiol groups[3] in enzymatic reactions. A unique property of the thiol group on enzymes is that the addition complexes it forms with a variety of compounds can be converted to even more reactive compounds in which cleavage of the covalent bond between sulfur of enzyme and carbon from substrate may provide the required energy for synthesis of C-C and C-N bonds as well as carboxyl phosphates and ordinary esters.

An extension of the concepts of Lynen and Racker, together with a consideration of certain type reactions which are well-documented in the organic chemical literature forms the basis of the presently described theory of fatty acid oxidation and carbon dioxide fixation. The proposed reaction sequences are largely speculative but are consonant with previously available data. During the past two years, experiments suggested by the theory have yielded results[4–6] with which the proposed reaction mechanism and predicted properties of the enzymes involved are consistent.

Any theory for the oxidation and synthesis of fatty acids must account for several generally accepted phenomena. First among these is the concept of *β oxidation* of fatty acids to yield two carbon fragments. This was established by Knoop[7] with confirming evidence from isotopic experiments.[8] Knoop's experiments also proved that *oxidative degradation proceeds from the carboxyl end of the fatty acid chain* and recent work[9–11] demonstrated that *fatty acid synthesis occurs by addition of two carbon units to the carboxyl end of the shorter fatty acids*. Finally the theory should account for the inability to detect intermediate breakdown products between the original fatty acid and acetoacetic acid.

Oxidation of Normal Fatty Acids.—The postulated mechanism for fatty acid oxidation is described in figure 1. The fatty acid oxidase (hereafter abbreviated FAO) is presumed to be an enzyme with two (or more) thiol groups (I in figure 1). The thiol groups may be similar or equivalent to one another with respect to their metabolic activity. The thiol groups may be those of two molecules of coenzyme A (case 1) or, alternatively, they may be those of two cysteine moieties in the protein molecule (case 2). Lipoic acid in its reduced form might provide the proper structure but there is as yet no evidence that this factor participates in fat oxidation or synthesis. In the event that the two sulfur atoms are not attached to closely adjacent carbon atoms, it is necessary for steric reasons to assume that the chains to which they are attached are

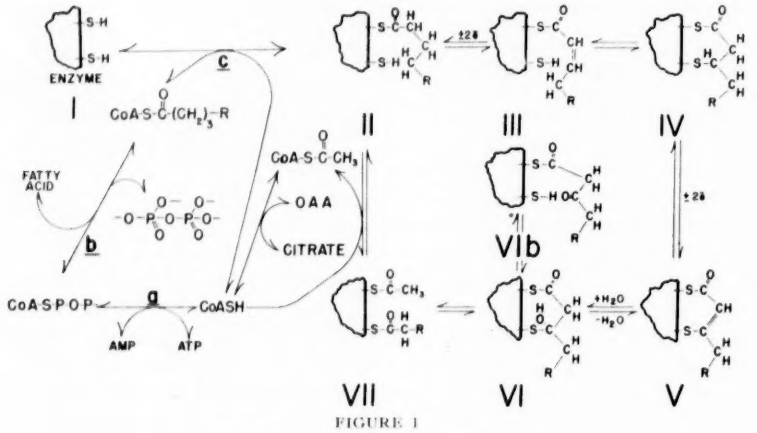


FIGURE 1

Proposed scheme for fatty acid oxidation.

held in close association by chemical binding. FAO in dried cells of *Clostridium kluyveri*[12, 13] and in the soluble extracts of rat liver mitochondria[5] is extremely sensitive to air and this would be expected of a multithiol enzyme.

Fatty acids must be activated, presumably by a reaction involving ATP before oxidation is possible by either particulate preparations[14] or soluble fatty acid oxidase.[5] The activation of acetate has been reported to involve formation of pyrophosphoryl coenzyme A,[15, 16] which in turn reacts with acetate to form acetyl-coenzyme A. In figure 1 an analogous scheme for activation of higher fatty acids is shown. It is assumed that the acyl coenzyme A may either combine with FAO to form II (case 1) or, if the second alternative mentioned above is true (case 2), the acyl group may be

transferred to a thiol group on the enzyme to form II. The activating enzyme has been differentiated from the oxidizing activity[5] indicating that at least two enzymes are involved in the over-all process. If a transfer of an acyl group from CoA to a thiol group on FAO is involved, there is adequate precedent for this type of exchange in the chemical literature [17–19] to assure its being possible by enzymatic catalysis.

FAO may then dehydrogenate the fatty acid at the α and β carbon atoms, yielding III which is assumed to be formed directly by reaction *c* when α,β-unsaturated fatty acids are provided as substrates. In animal tissues, α,β dehydrogenation seems to occur *in preference* to β,γ dehydrogenation[20, 21] but the data of Morehouse[20] indicate that some β,γ dehydrogenation does occur during butyrate oxidation in the rat. Vinyl acetic acid is metabolized by animal tissues[22] and also in dried cells of *Cl. kluyveri*;[23] both *cis* and *trans* crotonate are inactive in the latter[23] but are oxidized by animal tissues.[22, 24] These differences may depend upon specificity of the activating enzyme, *c*. Observations of Fisher-Hirschfelder models with a fixed carboxyl group show that the β and γ hydrogens can be moved into areas occupiable by α and β hydrogens. These observations support the contention (see below) that the branched chain acid—isovaleric—may be dehydrogenated in one of the β,γ positions. Up to this point, the sequence of reactions is similar to that proposed by Barker[25] for butyrate oxidation by *Cl. kluyveri*.

The second thiol group may now add spontaneously to the α,β-unsaturated ester (III) by a reaction which occurs at ordinary temperatures and which is readily reversible.[26–31] In such additions the sulfur becomes attached to the β carbon atom.[27, 28]

The dehydrogenation step may now be repeated on IV since available hydrogen atoms are again present on both α and β carbon atoms and presumably in juxtaposition with the dehydrogenation site. This theory predicts that dehydrogenation of the saturated fatty acid and the second oxidation to yield a compound at the oxidation level of keto acid are both effected by the same active group on FAO. The dehydrogenation may be linked to coenzyme I[5, 14, 16] or it may occur by a mechanism similar to that of succinic dehydrogenase where thiol groups and possible thiyl radicals [32, 33] are involved. If more than two thiol groups are present at the active site (see above) it is possible that they function both in the desmolysis of the carbon chain and in electron transport.

The unsaturated intermediate, V, may now add water to form the unstable VI which may rearrange to VII. The reversible interconversion of V and VI by addition or loss of water is analogous to the behavior of the compound formed by cysteine addition to $\Delta^{\beta,\gamma}$-angelicalactone,[34] and to the reversible dehydration of aldol.[34] When β-keto acids are being metabolized[35] it is assumed that they would be activated by reaction *c* to form

VIb directly. VIb and VI are in equilibrium; in the latter, the carbonyl group would tend to weaken the bond between the α and β carbon atoms. Migration of a proton to the α carbon atom would complete the cleavage.

From VII, the acetyl-coenzyme A may now dissociate (case 1) or the thioacetyl group may undergo transacetylation with coenzyme A (case 2). Acetyl-coenzyme A can react with a variety of enzyme systems but the bulk of the two carbon fragments are condensed with oxalacetate[36] and combusted in the Krebs tricarboxylic acid cycle. The fatty acid has been shortened by two carbon atoms but it remains combined, by a thiol ester linkage, with the enzyme (II) and another excursion of the cycle may proceed without further activation by ATP. Experimentally it has been found that only one mole of ATP is required to activate each mole of fatty acid.[5] When the fatty acid has been oxidized to the terminal 4 carbon atoms the last two thioacetyl groups are transferred to leave free FAO which must again be provided with an ATP-activated fatty acid.[5]

The depicted reaction scheme is one which could be readily reversed to account for fat synthesis from two carbon units. Separate enzymatic pathways for the synthesis and breakdown of fatty acids do not appear necessary. Activation of free acetate to yield acetyl CoA or oxidation of pyruvate to acetyl CoA would provide the necessary intermediate for fixing two carbon units to the enzyme. Suitable electron donors would be required for reduction of the two olefinic intermediate states. Such donors should become available in the cell when "high-energy" phosphate compounds are at a high concentration and respiration is limited by the lack of available phosphate acceptors.[37, 38] Steric factors may be responsible for stopping the synthesis when the carbon chains reach the length characteristic of the organ and species. The thioester linkage between the long chain fatty acids and FAO is a logical source of energy for the synthesis of glycerides and the carboxylic ester linkages in phospholipids.[39]

Acetoacetate Formation.—When acetyl groups on coenzyme A are not disposed of by condensation with oxalacetate, they condense with one another to yield acetoacetate. It has not yet been established whether the enzyme which catalyzes this condensation[40] is distinct from FAO. Nor is it apparent how the enzyme orients the acetyl units in such a manner as to convert C-1 labeled fatty acids to acetoacetate labeled unequally in C-1 and C-3. In animal tissues the penultimate carbon of fatty acids containing an even number of carbon atoms has a strong tendency to appear in the carbonyl group of acetoacetate.[41, 42] This could be explained if VI were converted to VIb more rapidly than to VII for then the terminal 4-carbon units (methyl end) would leave the enzyme at stage VIb without scission to two carbon fragments. A similar situation is encountered in extracts of *Cl. kluyveri* which oxidize 1-C^{14} butyrate in the absence of phosphate to acetoacetate containing C^{14} in the carboxyl group only.[13] At higher pH

however very little acetoacetate and much free acetate are formed[13] indicating a diversion of the type VI → VII followed by cleavage to free acetate. This is probably favored at high pH because the rate of hydrolysis of thioesters in alkali is four orders of magnitude greater than in acid.[43]

Carbon dioxide fixation into acetoacetate[44–46] will be discussed in the following section.

Oxidation of Branched Chain Acids.—The basic information concerning this subject has been discussed by Carter.[21] α-Methyl fatty acids are presumably cleaved between the α and β carbon atoms of the main chain

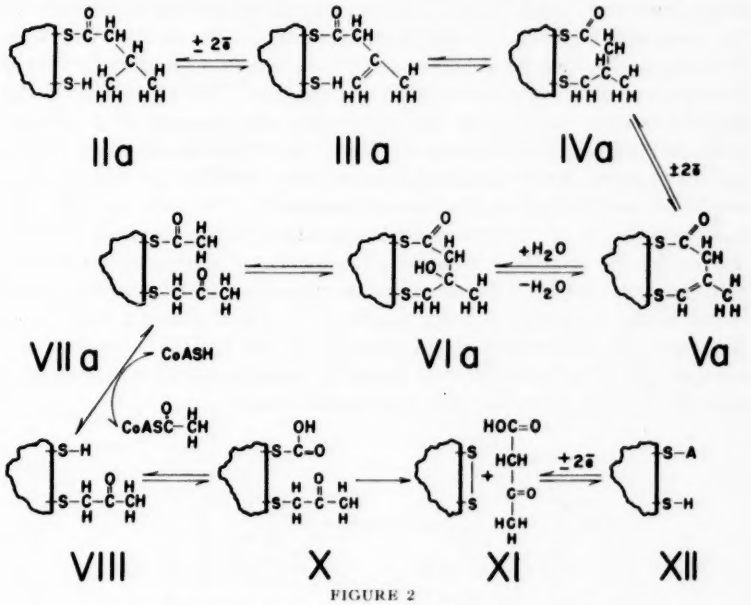


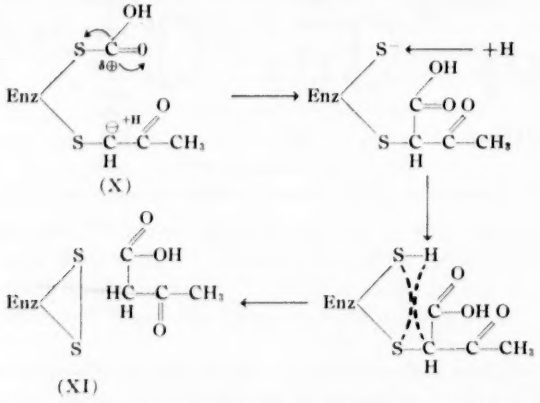
FIGURE 2

Proposed scheme for oxidation of branched chain fatty acids.

to yield propionate as the first cleavage product[47] (however, α-methyl caprylic is excreted without breakdown[48]). They could thus be metabolized by the scheme shown in figure 1. Fatty acids with groups larger than methyl in the α position are not readily attacked[48, 49] presumably because of steric hindrance.

The oxidation of β-methyl fatty acids involves rather unusual reactions. β-Phenyl β-methyl propionate is dehydrogenated in the α,β position and, presumably because this unsaturated acid resists further attack, it is excreted in the urine.[21, 50] Isovalerate is oxidized to yield its carboxyl and α-carbon atom as active acetate[45] while the isopropyl group is converted to acetoacetate by a stoichiometric fixation of carbon dioxide.[45, 46]

A hypothetical mechanism consistent with these findings is presented in figure 2. It is assumed that activation of the branched chain fatty acids occurs in the same manner as for straight chain acids, but that dehydrogenation of IIa occurs at the β,γ positions (see discussion above) to form IIIa. The second thiol group would add to the double bond with the sulfur attaching to the carbon atom farthest from the carbonyl group[30, 31] yielding IVa. Repeating the dehydrogenation step yields Va. α,β dehydrogenation at this step would serve the requirements of the theory equally well. It is apparent that oxidation of β-phenyl β-methyl propionate beyond the stage of its excretion product would not be possible by the proposed mechanism. Addition of water to Va would yield VIa. Cleavage of the bond between the α and β carbon atoms leading to VIIa is analogous to the reversal of an aldol condensation. The plausibility of the reaction is also enhanced by the observation that cleavage of β-hydroxy acids occurs under the influence of alkali,[51] heat[52] or acids.[53] From VIIa the acetyl group, which is derived from carbons 1 and 2 of the original acid, would be metabolized in the normal manner.[45] The three-carbon fragment attached to the enzyme in VIII is presumed to be converted to acetoacetate by the remaining sequence of reactions. Carbonic acid may possibly be activated by ATP as are carboxylic acids. The possible role of biotin in this activation is being sought.[44, 54] The C-C and S-S bonds of XI would be synthesized at the expense of the two C-S bonds. The reaction should be possible from energetic considerations[55] and may proceed through the following ionic intermediate steps.



The reduction of the S-S form of the enzyme to the free SH form and possible simultaneous fixation of acid will be discussed below.

Incorporation of isotopic bicarbonate into acetoacetate from normal fatty acids,[46] pyruvate[46] and acetyl phosphate[5] may possibly involve an

analogous reaction sequence. If the second thiol group of FAO were to add to an enolized form of VIb with the sulfur attaching to the methine carbon, hydrolytic cleavage followed by exchange of the carboxyl with subsequent condensation would yield FAO in the disulfide form (XI) and β-hydroxybutryate, the reoxidation of which would yield labeled acetoacetate. Experiments are now under way to determine whether acetoacetate or β-hydroxybutyrate is the primary labeled product.

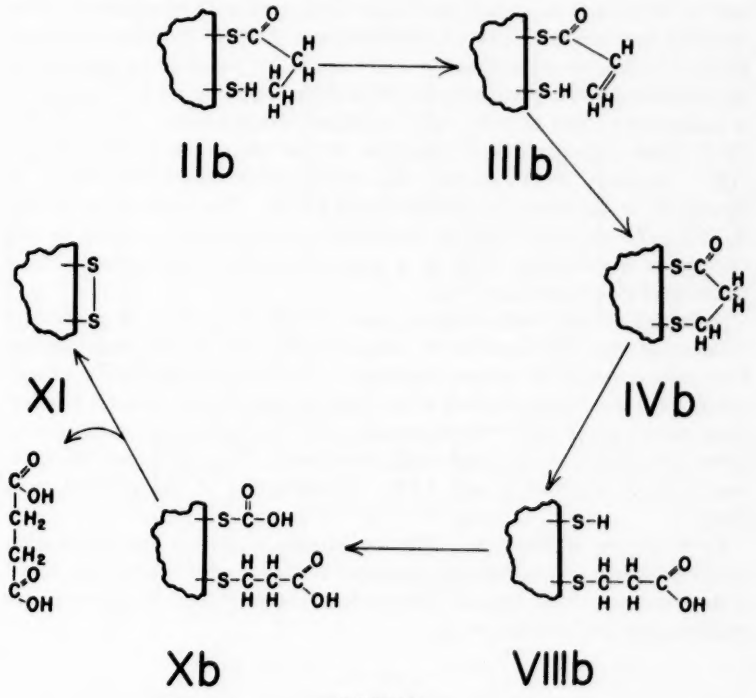


FIGURE 3

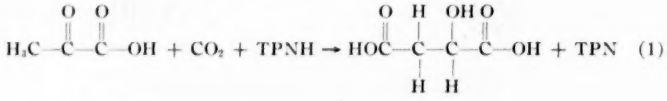
Proposed scheme for propionate metabolism.

The Pathway of Propionate Metabolism.—It has been known for some time that propionate is a glycogenic substance for the rat. Experiments with isotopically labeled propionate[56] indicated that the α and β carbon atoms of this fatty acid are randomized before being converted to carbohydrate-like materials. Recently Shreeve[57] demonstrated that complete randomization of these two carbons occurred also during the formation of acetate from propionate, and Daus, Meinke and Calvin[58] demonstrated

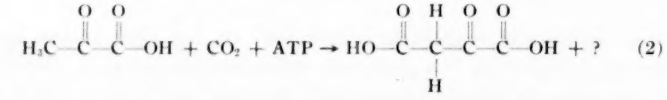
their complete randomization in the conversion of propionate to lactate. Both groups of investigators concluded that propionate is either converted to a symmetrical compound such as succinate before it is oxidized to pyruvate or that the direct oxidation to pyruvate is in some unknown manner accompanied by an extremely rapid equilibration with a symmetrical compound. Both groups of workers also provided evidence for the activation of propionate—Shreeve[57] by the detection of N-propionylphenyl amino butyric acid and Daus, *et al.*,[58] by the detection of β-hydroxy-valeric acid in filtrates from mouse liver slices incubated with propionate. This evidence together with that of Stadtman and Barker[59] indicates that propionate undergoes activation like fatty acids. A pathway for propionate metabolism consistent with the above is described in figure 3. Activation is analogous to that of other fatty acids and the substrate combined with FAO could undergo dehydrogenation to the thioacrylate intermediate IIIb. Addition of the second thiol group would yield IVb which on hydrolysis of the thioester linkage forms VIIIb. The latter is an analog of VIII in figure 2, and could be converted by CO_2 fixation to succinate and enzyme in the disulfide form by a series of reactions analogous to those described for acetoacetate.

Although no published evidence exists for the conversion of propionate to succinate by CO_2 fixation in animal tissues, the reverse reaction has been demonstrated in certain bacteria.[60] To test the hypothesis, experiments with the soluble extract of rat liver mitochondrial acetone powder[5] have been carried out. Incorporation of $C^{14}O_2$ under anaerobic conditions into organic, non-volatile acids was found dependent upon the presence of both propionate and ATP. Identification of the product is in progress.

Carboxylation of Pyruvate.—The mechanism of carbon dioxide fixation described above may be logically extended to the "malic" enzyme reaction[61] (equation 1), and to the ATP-dependent carboxylation of pyruvate to yield oxalacetate[62] (equation 2).



(1)



(2)

The "malic" enzyme may conceivably have two thiol groups, a group possessing affinity for the carboxyl group of pyruvate (labeled X in figure 4), and a site which binds Mg^{++} or Mn^{++}. The reaction by which CO_2 is esterified with one of the thiol groups to yield I will be discussed below.

Pyruvate as the enol is assumed to combine with group X and with Mn^+— as in II.[63] Addition of a thiol group to the enol would occur spontaneously with sulfur attaching to the methylene carbon.[28] The enol structure having been destroyed, the hydroxyl would lose its strongly acidic character and combine with a proton as in IV. It is likely that the enzyme orients the addition of H to the α carbon atom in such a manner as to yield a substituted lactic acid of L-configuration. Carboxylation of the β carbon atom by the scheme outlined above would yield L-malic acid[64] and enzyme

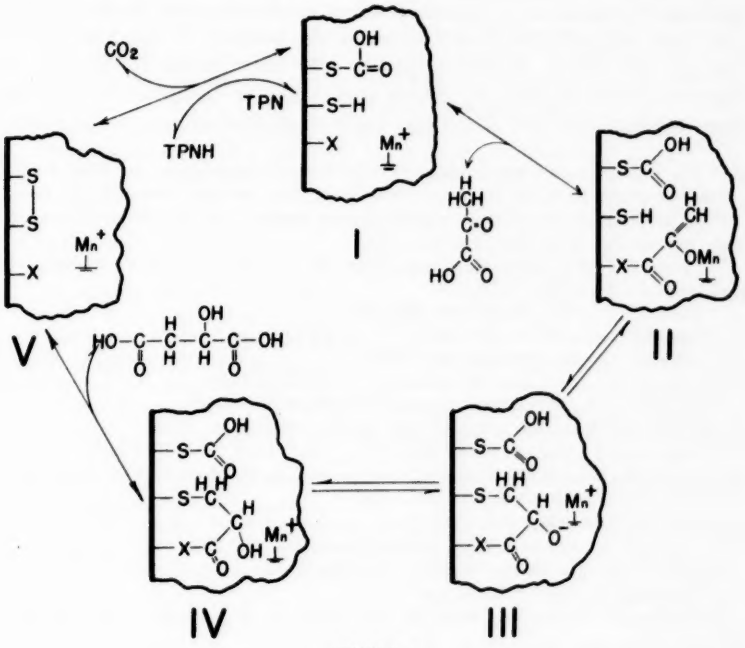


FIGURE 4

Hypothetical scheme for the "malic" enzyme reaction.

in the disulfide form. It is assumed that reduction of the disulfide occurs by reaction with TPNH (compare reduction of G-S-S-G by TPNH[65–67] and of cystine by DPNH[68]) and that the fixation of carbon dioxide is coupled with this reduction. A possible mechanism is as follows. The disulfide may be reduced to one thiol and one thiyl radical by a single electron transfer. The thiyl radical may then combine with $\cdot C(=O)-R$ while the

second electron from TPNH may accomplish the reduction of HO· to HO^-. Such a scheme would provide for the continuous fixation of CO_2 at the expense of TPHN. It is possible that oxidative phosphorylation involves a similar fixation of phosphoric acid during reduction of disulfide groups in the electron transport enzymes. Reversibility of the reactions in figure 4 would account for the malic dehydrogenase activity of the purified enzyme.[61] It would be predicted that the enzyme would be sensitive to *p*-chloromercuribenzoate and this has been found to be the case.[6]

Utter and Wood found that ATP is necessary for CO_2 fixation into oxalacetate by pigeon liver preparations containing oxalacetic decarboxylase[62] and that reduced glutathione enhances the amount of CO_2 fixed in the presence of ATP.[69] In this case, too, ATP may furnish the energy for fixation of CO_2 in amounts greater than would be permitted by equilibrium of reaction 2 and it appears likely that thiol groups are involved.

* The experimental work designed to test the hypotheses herein described is supported by grants from the Nutrition Foundation, Inc., and the United States Public Health Service. Presented at the annual meeting sponsored by the Nutrition Foundation, Skytop, Pa., April 1952.

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AUXIN-INDUCED GROWTH INHIBITION A NATURAL CONSEQUENCE OF TWO-POINT ATTACHMENT*

By Robert J. Foster, D. Harold McRae, and James Bonner

Kerckhoff Laboratories of Biology, California Institute of Technology

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Introduction.—It is characteristic of a great number of biologically active substances that the responses which they elicit are twofold, low concentrations of the material promoting a particular activity, and higher concentrations inhibiting it. This is the case with the auxin-induced growth responses of plants. An active auxin such as indole acetic acid (IAA) brings about and is essential to growth in length of stems, hypocotyls and other plant organs including the Avena coleoptile. Excised sections of this object grow in length when they are floated in solutions containing low concentrations of IAA and this growth is a simple function of auxin concentration as

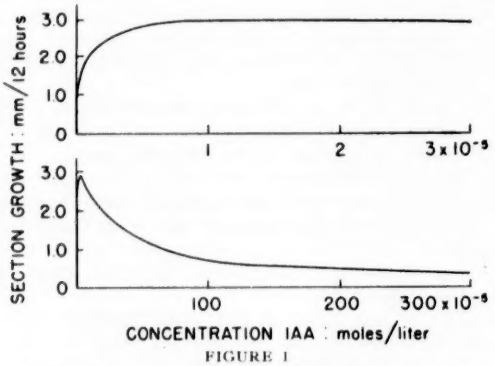

FIGURE 1

Avena coleoptile section growth as a function of IAA concentration: *a* (above), over a low concentrations range $0\text{–}3 \times 10^{-5} M$; *b*, over a greater concentration range $0\text{–}300 \times 10^{-5} M$.

is shown in figure 1a. With increasing auxin concentration, however, growth rises to a maximum only to decline as auxin concentration is still further increased (Fig. 1b).

It has been shown in earlier papers[1] that a molecule in order to be active as an auxin must satisfy a number of structural requirements among which is the requirement for two suitably positioned reactive groups. These two groups, a carboxyl group located in a side chain attached to an unsaturated cyclic nucleus[2] and a substitutable group of critical reactivity in the nucleus ortho to the side chain[3] appear to be the functional groups through which

the auxin molecule makes attachment to some receptor entity within the plant. The combined auxin, bound through its two reactive positions to two functional sites of the receptor entity, appears to be the form which is active in eliciting growth responses. Although the exact chemical nature and biochemical function of the auxin-receptor complex is not known, the existence and character of the complex can be deduced from kinetic considerations[4] and from the study of the relationships of chemical structure to physiological activity of synthetic auxin analogs.[5]

It is the purpose of this paper to show that the inhibition of growth induced by supra-optimal concentrations of auxin is a natural and indeed an inescapable consequence of the two-point attachment by which auxin is bound within the plant to form the functional entity.

Materials and Methods.—All of the experimental work reported here has been based on measurements of the rate of elongation of excised sections of Avena coleoptiles floated in solutions containing an active auxin.[6] Avena seedlings were grown in vermiculite contained in stainless steel trays and maintained in a dark room under low intensity red light at a temperature of 25–26°C. and a relative humidity of 90%. When the seedlings had attained an age of 82–84 hrs. they were selected for uniformity. Those plants with a coleoptile length between 2.75 and 3.25 cm. were decapitated by removal of a 2–3 mm. apical tip and the first leaves removed. Five mm. apical sections were then cut from each coleoptile with a double-bladed cutting tool. The sections were pooled and randomly distributed in lots of 20 to individual Petri dishes containing 20 ml. of the test solution. A basal medium containing 0.0025 *M* potassium maleate buffer (pH 4.5) and 3 per cent sucrose was employed. To this the varied auxins, adjusted to pH 4.5, were added as desired.

Sections were allowed to grow in the dark at 25–26° for a period of 12 hrs. and their length then measured under a microscope. This growth measurement yields the initial growth velocity since the growth of Avena coleoptile sections under the present conditions is linear with time for a period of 18 hrs. or more.

Kinetic Treatment of Avena Section Growth.—The elongation of excised *Avena coleoptile* sections in response to exogenous auxin may be formally treated by the enzyme kinetics of Michaelis and Menten.[7] For this treatment, the initial growth velocity of the sections (v) is determined as a function of substrate (IAA) concentration [S]. Classical enzyme kinetics are based upon the hypothesis that substrate combines with enzyme (E) to form an enzyme-substrate complex (ES) which then decomposes to form the product of the reaction (in this case growth) and to reform E. In this formulation which is outlined in equation (1), K_s and k are constants which express, respectively, the affinity constant for the formation of ES from E

$$E + S \underset{}{\overset{K_s}{\rightleftharpoons}} ES \xrightarrow{k} \text{growth} + E \qquad (1)$$

and S and the constant characterizing the rate of decomposition of ES to end product.

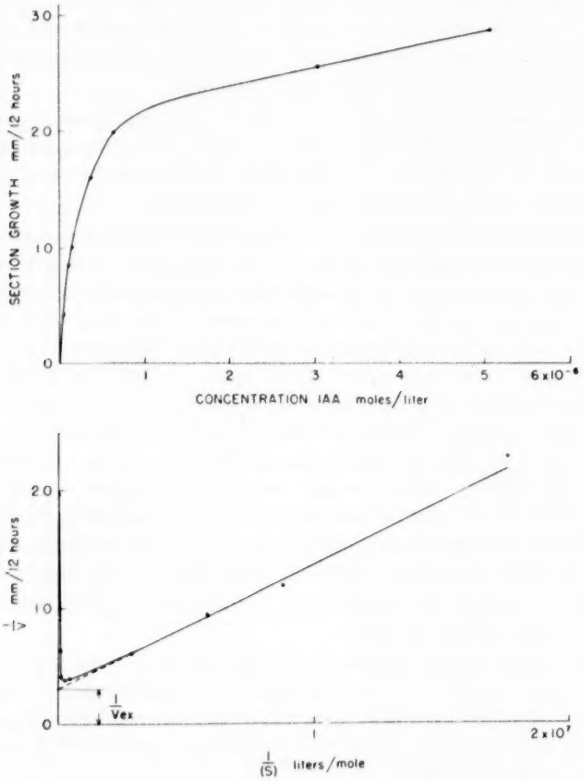

FIGURE 2

Avena coleoptile section growth as a function of IAA concentration: *a* (above), growth rate plotted as a function of IAA concentration; *b*, reciprocal of growth rate plotted as function of reciprocal of concentration.

Equation (1) may be expressed in terms of reaction velocity (v) which is dependent, as shown in equation (2) upon $[S]$ and the constants K_s and

$$v = \frac{V_{max}\,[S]}{K_s + [S]} \qquad (2)$$

$V_{max.}$ The latter is determined as the reaction velocity of the system in the presence of non-limiting substrate concentration and in which all E is therefore combined as ES. The relationship between v and $[S]$ for any system which follows the formulation of equation (2) must be expected to be a hyperbolic one in which $V_{max.}$ represents the value of v which is asymptotically approached as $[S]$ is increased and in which K_s is the value of $[S]$ at which one half of the maximum reaction velocity is attained. Figure 2a shows how the data relating growth rate of Avena coleoptile sections to IAA concentration fall smoothly on such an hyperbola.

A critical test of the applicability of equation (2) to a particular system consists in relating graphically the reciprocal of reaction velocity ($1/v$) to the reciprocal of substrate concentration ($1/[S]$). Any system following Michaelis-Menten kinetics is expected to yield a straight line for which the intercept on the ordinate is $1/V_{max.}$ and the slope $K_s/V_{max.}$. Figure 2b shows that the data for Avena section growth as a function of IAA concentration fit this formulation with precision over the lower range of substrate concentration. The systematic deviation of the data for the higher concentrations, which is an expression of a depression in growth rate at the higher auxin levels, may be understood in the light of an extension of the Michaelis-Menten treatment as will be shown hereafter.

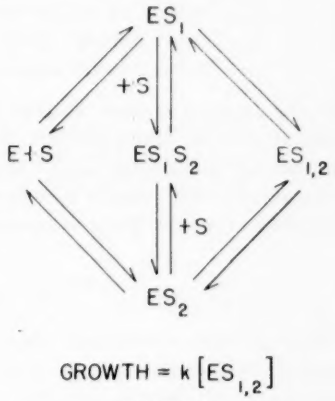

FIGURE 3

Equilibria relating the formation of active receptor-substrate complex ($ES_{1,2}$) to the formation of inactive receptor-substrate complex (ES_1S_2).

Kinetic Treatment of Two-Point Attachment.—The treatment thus far has dealt implicitly with systems in which attachment of substrate to enzyme is consummated through a single point. Let us now consider the manner in which two-point attachment may be expected to alter this picture.[8] A molecule of auxin (S) might be expected to become first associated with one of the two receptive sites on the enzymatic entity and then, by becoming associated with the second receptive site, to consummate the two-point attachment essential to enzymatic (growth) activity. This sequence of reactions will predominate at low auxin concentrations. At higher auxin concentrations, however, there is an appreciable probability that two auxin molecules may simultaneously approach the receptor entity and simultaneously react with it, each combining through a single reactive group. Such interaction will yield an inactive auxin-receptor complex in which

each auxin molecule prevents the other from consummating a two-point attachment.

The several equilibria which relate the monomolecular and bimolecular substrate-receptor complexes are illustrated in figure 3. These equilibria involve the following quantities:

E = Receptive entity of the plant.
S = Auxin.
ES_1 = Auxin-receptor entity complex combined through point 1.
$ES_{1,2}$ = Auxin-receptor entity complex combined through points 1 and 2. This complex, the enzymatically active one, may be presumed to be formed from either ES_1 or ES_2.
ES_1S_2 = Bimolecular auxin-receptor entity complex in which two auxin molecules are involved, one attached through point 1, the other through point 2. This complex is presumed to be enzymatically inactive.

In the classical enzyme kinetics only the equilibria which result in the formation of the active complex, $ES_{1,2}$ are considered. If we include in the consideration the formation of the complex ES_1S_2, the classical expression relating reaction velocity to substrate concentration may be shown to be altered to the form given in equation (3).

$$v = \frac{V_{ex}\,[S]}{K_s' + [S] + [S]^2/C} \qquad (3)$$

This expression differs from that of classical kinetics in that it includes a term $[S]^2/C$ which is a measure of the probability that a second molecule of substrate will become attached to the receptor entity before the first molecule has consummated its two-point attachment. The reaction velocity v, which is proportional to $ES_{1,2}$, is expressed in terms of $[S]$ and the experimentally determinable constants V_{ex}, K_s' and C. Equation (3) predicts a maximum reaction velocity (growth rate) at a substrate concentration of $\sqrt{K_s' \cdot C}$. This maximal rate is decreased below the rate which would be achieved in a classical system by the amount to which the inactive complex ES_1S_2 is formed in the system at the substrate concentration $\sqrt{K_s' \cdot C}$.

Equation (3) will now be applied to data derived from experimental determination of the relation of auxin concentration to rate of Avena coleoptile section growth. We shall first evaluate the constants V_{ex} and K_s' which correspond to the constants $V_{max.}$ and K_s of classical enzyme kinetics. These two constants may be determined in the low range of auxin concentration in which the formation of the bimolecular complex ES_1S_2 is negligible. This is most readily done on the reciprocal plot of figure 2b. V_{ex} is the extrapolated intercept on the ordinate while K_s' is the slope

divided by the intercept. The remaining constant C, which is a measure of the probability that two molecules will react with the two receptor sites of the reactive entity before either has had a chance to consummate its two-point attachment, must be evaluated in the region of high concentration in which the formation of this complex predominates. This is done as

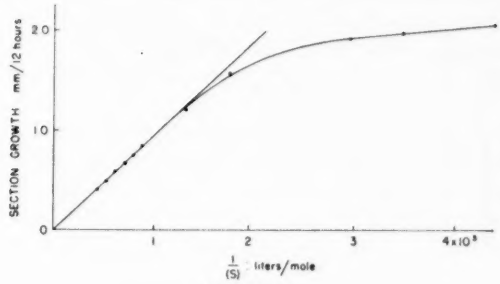

FIGURE 4

At high auxin concentration, growth of Avena coleoptile sections is inversely proportional to the substrate concentration. The limiting slope of the lower portion of this curve = $V_{ex} \cdot C$.

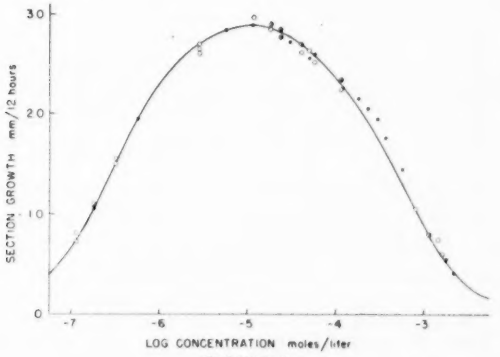

FIGURE 5

Avena section growth as a function of IAA concentration. The solid line is that for equation (3); the points represent data from five separate experiments.

shown in figure 4 which presents growth velocity as a function of the reciprocal of the substrate concentration. It can be seen that as the auxin concentration is increased, the growth velocity becomes inversely proportional to it and from the limiting slope which is $V_{ex}C$, the value of C may be determined.

It appears then that at the extremes of the auxin concentration range auxin affects growth in the manner predicted by equation (3). This equation describes accurately the effects of auxin on the growth response of Avena coleoptile sections over the entire range of auxin concentrations to which this organ is sensitive, a range of one hundred thousand fold. That this is so is shown in figure 5. Since the range of concentration is so large,

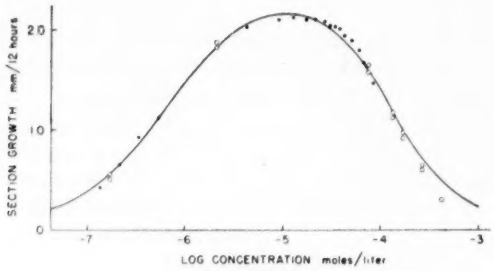

FIGURE 6

Avena section growth as a function of 2,4-D concentration. The solid line is that for equation (3); the points represent data from two separate experiments.

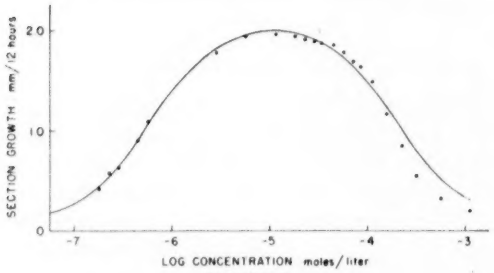

FIGURE 7

Avena section growth as a function of NAA concentration. The solid line is that for equation (3); the points represent experimental data.

the data are presented in the form of initial growth velocity as a function of the log of auxin concentration. The curve of figure 5 is that for equation (3) using the values for the constants determined as outlined above. The individual points of figure 5 represent data from 5 separate and complete experiments. These fit the curve within experimental error over the entire 100,000 fold range of concentrations. That the maximum growth rate occurs at a substrate concentration of $\sqrt{K_s' \cdot C}$ is also verified by the data.

The synthetic growth substances such as 2,4-dichlorphenoxy acetic acid (2,4-D) and naphthalene acetic acid (NAA) are, like the naturally occurring IAA, active in promoting the growth of Avena coleoptile sections at low concentrations, active in inhibiting growth at higher concentrations. That the kinetic relations of 2,4-D and of NAA in their effects on Avena section growth are also in accord with the two-point attachment concept is evident from the data of figures 6 and 7. This is true even though the constants V_{ex}, K_s' and C for these substances differ appreciably among themselves and differ also from those determined for IAA, a fact summarized in table 1.

Discussion.—The concept that auxins are bound within the plant to an appropriate receptor entity through which auxin-induced growth responses are mediated was originally based not only on the known binding of auxins to plant proteins[9] but also and more specifically on the fact that the kinetics of auxin action are in close accord with the expectations of classical

TABLE 1

SUMMARY OF THE PARAMETERS WHICH RELATE GROWTH OF AVENA COLEOPTILE SECTIONS TO AUXIN CONCENTRATION

Auxin	V_{ex}, mm./12 hrs.	K_s',[a] moles/liter	C,[b] moles/liter	$\sqrt{K_s' \times C}$,[c] moles/liter
IAA	3.1	3.4×10^{-7}	4.0×10^{-4}	1.2×10^{-5}
2,4-D	2.3	5.5×10^{-7}	1.4×10^{-4}	0.9×10^{-5}
NAA	2.1	6.4×10^{-7}	1.7×10^{-4}	1.1×10^{-5}

[a] Concentration of auxin at which growth rate is 0.5 V_{ex} and $ES_{1,2} = E_{free}$.

[b] Concentration of auxin at which growth rate is 0.5 maximal due to formation of the complex which contains two auxin molecules. $ES_{1,2} = ES_1S_2$.

[c] Concentration of auxin at which growth rate is maximal.

enzyme kinetics.[4] The extension of this concept to include the specification that the binding must be consummated through a two-point attachment of the auxin molecule to the receptor entity was first suggested by the observation of Muir, Hansch and Gallup[3] that a free ortho group of critical reactivity as well as a free carboxyl group are essential to auxin activity. The hypothesis that both of these groups are in fact essential to auxin activity and by implication essential to auxin binding has been strengthened by the finding that auxin-like molecules in which either of the two essential groupings is blocked or abolished are not only inactive as auxins but also are in fact competitive inhibitors of auxin action.[10] The matters considered in this paper constitute a critical experimental test of the two-point attachment concept as applied to auxin action. If two-point attachment of auxin to its receptor is essential for the formation of the complex which induces growth it follows that inhibition of growth by the formation of bimolecular complexes may be expected at higher auxin concentrations. The fact that auxin-induced growth inhibition not only does occur but also

that it follows in elegant detail the kinetics of bimocular complex formation provides strong support of the whole two-point attachment concept.

Of what practical use may it be to express and to think about auxin-induced growth inhibition in terms of the kinetics of two-point attachment? One evident application may be in the study of the herbicidal action of synthetic growth substances such as 2,4-D. It is a peculiarity of these materials that although they are exceedingly deleterious to particular plants at higher concentrations, they act at lower concentrations in a manner indistinguishable from that of IAA itself. We might therefore inquire in how far the herbicidal action of 2,4-D and related materials may be owing to the formation of bimolecular complexes of the kind here described and we might attempt to answer this question by determination of the extent to which herbicidal activity is a function of the second power rather than of the first power of the applied growth substance concentration.

The twofold action of auxin on plant growth appears then to have its basis in a requirement for two-point attachment of the auxin to its receptor, a requirement which makes possible the formation of an inactive bimolecular complex at higher auxin concentrations. It will be of evident interest to discover whether the twofold action of other biologically active substances may in some instances similarly reside in a requirement for multiple attachment.

Summary.—The relationship between Avena coleoptile section growth rate and auxin concentration over a wide range has been demonstrated to be predictable on the basis of a requirement for two-point attachment of the auxin molecule to some receptive entity in the plant. In particular, the inhibition of growth at higher auxin concentrations, which may be the cause of herbicidal activity of certain chemicals, has been shown to be a natural and predictable consequence of the two-point attachment of the auxin molecule to the receptor entity within the plant.

* Report of work supported in part by the Herman Frasch Foundation.

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CONDUCTANCE OF SOLUTIONS OF SALTS IN BROMINE

BY GUSTAVE W. MOESSEN AND CHARLES A. KRAUS

METCALF RESEARCH LABORATORY, BROWN UNIVERSITY

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According to Darby,[1] trimethylammonium chloride is readily soluble in bromine at ordinary temperatures. The concentrated solutions are very viscous and are transparent with a pale yellow color. At 18°C. and a concentration of 0.029 N, the equivalent conductance, Λ, is only 0.0253; at a concentration of 1.236 N, the conductance is 11.5. The increase of conductance with increasing concentration is a very general phenomenon in solvents of low dielectric constant. However, the high value of the conductance in bromine is exceptional. In view of these circumstances it appeared of interest to investigate the properties of solutions of salts in bromine somewhat further.

We have measured the conductance of solutions of tetra-*n*-butylammonium bromide from approximately 0.01 to 1.3 N and that of trimethylammonium chloride from 0.03 to 3.6 N at 25°C.

Measurements were made with alternating current in the audio frequency range. For the concentrated solutions a Jones bridge was employed and, for the dilute solutions of high resistance, a parallel arm bridge capable of measuring 100 megohms with precision. Two conductance cells were employed having constants of 0.6264 and 3.008. They had a volume of 100 and 150 cc., respectively. The electrode chamber of the first cell was quite small so that only a comparatively small volume of solution was required. Concentrated solutions were made up by adding, successively, known weights of bromine to a known weight of salt, these weights being determined by weighing the cell and contents. More dilute solutions were made up by adding, successively, known weights of salt to a known weight of bromine.

Corrections were made for the weight of solvent in the vapor phase. The cells were placed in an oil thermostat whose temperature was controlled to 25 ± 0.002°.

Bromine was purified by a slight modification of the method of Terwogt[2] in which zinc oxide, prepared from zinc carbonate, was employed to eliminate iodine.[3] The specific conductance of the solvent was $1\text{–}6 \times 10^{-10}$ and was negligible in comparison with that of the solution in all cases. At 25°, the density of bromine is 3.102; its viscosity is 9.563×10^{-3} Poise; the dielectric constant is 3.147 and the vapor pressure is 214 mm.

In order to determine the concentration of the solutions on a volume basis, their densities were determined. Darby[1] had shown that the

density of solutions of trimethylammonium chloride is a linear function of molal concentration; accordingly, the density of one concentrated solution was measured for each salt. For tetrabutylammonium bromide, the density is 2.897 at 0.4471 *M;* for trimethylammonium chloride, it is 2.707 at 1.1025 *M*. Densities of various solutions were then found by graphical interpolation.

The salts were prepared and purified according to conventional methods. Tetrabutylammonium bromide melted at 112–113°; trimethylammonium chloride sublimed in the neighborhood of 226°.

The results of conductance measurements are presented in table 2 for tetra-*n*-butylammonium bromide and, in table 1, for trimethylammonium chloride. Concentrations, in moles per liter of solution, are given in

TABLE 1

CONDUCTANCE OF TRIMETHYLAMMONIUM CHLORIDE IN BROMINE AT 25°

C	Λ	C	Λ
3.641	12.43	0.6107	7.759
2.617	15.29	0.4433	5.579
2.011	15.75	0.03616	0.01217
1.299	13.58	0.1246	0.5184
1.035	11.82	0.1590	1.041
0.8338	10.11	0.2678	2.863

TABLE 2

CONDUCTANCE OF TETRABUTYLAMMONIUM BROMIDE IN BROMINE AT 25°

C	Λ	C	Λ
0.01188	0.05668	0.7156	12.14
0.02945	0.2470	0.6319	11.13
0.05525	0.6767	0.5585	10.13
0.09203	1.431	0.4522	8.449
0.1206	2.060	0.3846	7.288
0.1653	3.037	0.3056	5.828
1.298	12.77	0.2212	4.151
1.135	13.92	0.1641	2.957

column 1 and equivalent conductances, Λ, in column 2. The results are shown graphically in figure 1, where the logarithm of equivalent conductance is plotted as ordinate against the logarithm of concentration as abscissa.

Both salts are poor conductors at low concentrations; at 0.036 *N*, the conductance of the bromide is approximately 30 times that of the chloride. The conductance increases greatly with increasing concentration. The conductance of tetrabutylammonium bromide reaches a maximum of 13.92 at 1.135 *N;* the conductance of trimethylammonium chloride has a maximum of 15.75 at 2.011 *N*. The conductance decrease at concentrations above that of the maximum is due to the increasing viscosity of the solution.

The form of the conductance curve of these salts in bromine conforms to that of similar salts in solvents of low dielectric constant. However, the conductance at higher concentration is much greater than in other solvents. Considering the viscosity of the solvent and the markedly greater (but unknown) viscosity of the more concentrated solutions, it follows that these salts are highly ionized at these concentrations. In accord with observations in other solvents, dilute solutions of terabutylammonium bromide in bromine are much more highly dissociated into ions than solutions of trimethylammonium chloride, approximately 30 times at 0.036 *N*. At high concentrations, however, the two salts are dissociated to much the same extent. For the bromide, the maximum appears at lower concentration than for the chloride because of the greater viscosity of its solutions as a result of its much larger positive ion.

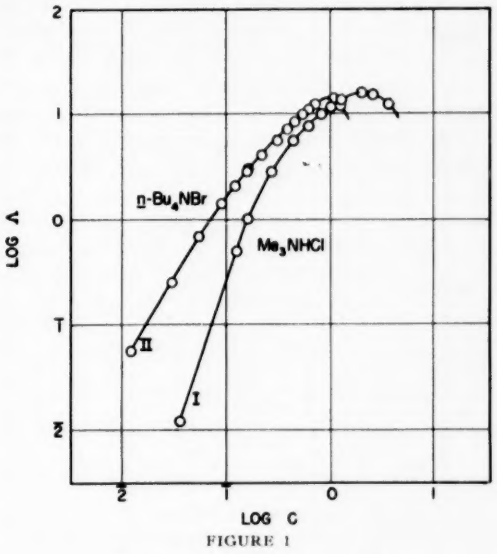

FIGURE 1

Conductance of salts in bromine.

The behavior of these salts in bromine is in accord with the view of Strong and Kraus,[4] that the more concentrated solutions of salts in solvents of lower dielectric constant are best looked upon as solutions of solvent in supercooled, fused salts. This view is strongly supported by the measurements of Seward[5] with solutions of tetrabutylammonium picrate in *n* butanol at 91°C. Seward measured the conductance and the viscosity of this system all the way from pure butanol to pure fused salt. The conductance-viscosity product was found to be the same for the pure fused salt as for an infinitely dilute solution. The conductance-viscosity product passes through a minimum at 0.08 *N*. Above a concentration of 0.75 *N*, the product increases almost linearly with composition up to that of the pure fused salt.

The conductance of an electrolyte, other things being equal, is nearly proportional to the fluidity (the reciprocal of viscosity) of the medium. The increase in the conductance-viscosity product is, to a first approximation, a measure of the increase in the number of free ions present.

Many years ago, Lewis and Wheeler[6] measured the conductance of potassium iodide in iodine at 140°C. This important investigation seems to have been overlooked, probably because concentrations and equivalent conductances were expressed in arbitrary units and could not be compared with those of other systems without carrying out the necessary computations. We have computed the concentrations and equivalent conductances of these solutions. In doing this, we have assumed the density of liquid iodine at 140°C. to be 3.7, that of potassium iodide, 2.9, and, further, that the volume of these solutions is equal to the sum of the volumes of solute and solvent. The values computed on this basis are not precise because the data for the densities of iodine and potassium iodide relate to temperatures differing somewhat from 140°. However, the error can hardly be greater than one per cent and does not affect any conclusions that we may draw.

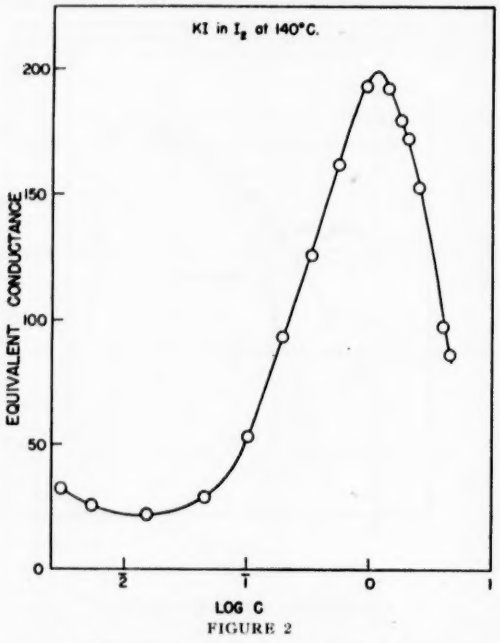

FIGURE 2

Conductance of potassium iodide in iodine.

The conductance of potassium iodide in iodine is shown graphically in figure 2, where equivalent conductances are plotted as ordinates and the logarithms of concentrations as abscissae. It will be noted that the conductance increases from a value of 22 at 0.013 N to 200 at 1.14 N. Above this concentration, the conductance decreases, reaching a value of 90 at 4.6 N. There appears to be a flat minimum at 0.013 N.

The conductance curve of potassium iodide in iodine is in most respects very similar to that of salts in bromine. However, the conductance is very much greater than it is for salts in the latter solvent. We do not know the value of the dielectric constant of liquid iodine, nor do we know its viscosity. The maximum in iodine occurs at practically the same concentration as in solutions of tetrabutylammonium bromide in bromine.

The viscosity of iodine must be of the same order of magnitude as that of bromine and the same holds for solutions in the two solvents. The concentrations at which the minimum conductance appears would seem to indicate a dielectric constant markedly higher than that of bromine at 25°. In benzene (D. const. = 2.28), the conductance minimum for a strong salt lies at 1×10^{-5} N; for anisole ($D = 4.29$), it lies at 1×10^{-3} and for ethylene bromide ($D = 4.76$), it lies at 5×10^{-3} N. On this basis, the dielectric constant of liquid iodine at 140°C. might be expected to lie in the neighborhood of 5.0.

In any case, we cannot escape the conclusion that potassium iodide in iodine solution is very measurably dissociated into free ions at low concentrations and highly dissociated at higher concentrations. If we may be permitted to repeat, to gain an understanding of concentrated electrolytic solutions, we shall have to consider them from the point of view of solutions of solvent in salt. In solvents of low dielectric constant, the state of the system is determined by the charged ions of the electrolyte; the lower the dielectric constant, the lower the concentration down to which the ions are the dominant factor.

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VELOCITY ULTRACENTRIFUGATION OF HETEROPOLY ACIDS

BY S. J. SINGER* AND JOHN L. T. WAUGH

GATES AND CRELLIN LABORATORIES OF CHEMISTRY,† CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA

Communicated by Linus Pauling, October 6, 1952

This communication is concerned with some preliminary velocity ultracentrifuge experiments which we have performed with 12-tungsto-silicic acid and some other heteropoly acid salts. It was suggested to us by Professor L. Pauling that because of their large densities these compounds might sediment at measurable rates in the ultracentrifuge despite their relatively low molecular weights. Upon investigation it was found that not only were the sedimentation constants (s) easily measurable, but diffusion constants (D) could be determined from the same sedimentation diagrams. In combination with partial specific volume ($\bar{V}$) determinations, it was possible to calculate molecular weights for the anions in question by means of the familiar Svedberg rate sedimentation equation:[1]

$M = RTs/D(1 - \bar{V}d)$. To our knowledge, no ionically dispersed inorganic salt has previously been studied in the velocity ultracentrifuge, and a molecular weight determined from combined measurements of its sedimentation and diffusion constants.

In order to determine the weights of the anions, the compounds were investigated in buffered solutions of fairly large ionic strengths. The

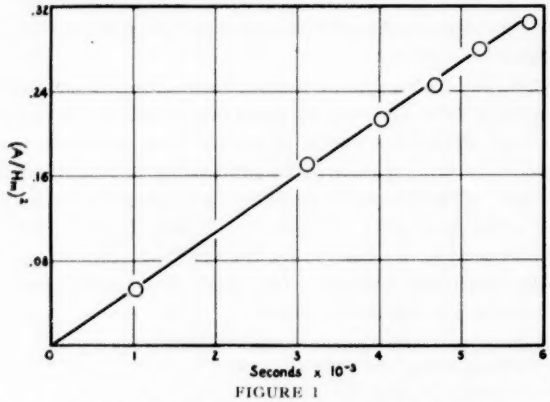

FIGURE 1

A plot of diffusion data obtained during the sedimentation of 12-tungsto-silicic acid.

TABLE 1

PHYSICAL CONSTANTS OF HETEROPOLY ACIDS AND SALTS[a]

COMPOUND[b]	BUFFER	SOLUTE CONC., G./100 ML.	$\bar{V}$	$s^{W}_{20} \times 10^{13}$, SEC.	$D^{W}_{20} \times 10^{6}$, CM.^{2}/SEC.	$M_{formula}$[c] HYD.	UNHYD.	Msed.
a	NaAc-HAc, pH 5.23, μ 0.2	1.20	0.2524	3.8	4.4	3307	2875	2800
b	NH_4Ac, 0.50%, pH 7.0	1.40	0.2846	3.5	3.5	6204	5340	3400
b	KH phthalate–HCl, pH 2.8, μ 0.1	1.39	0.2743	3.6	4.0	6204	5340	3000
c	KH phthalate–HCl, pH 2.8, μ 0.1	2.04	0.3225	3.4	4.1	2742	2346	3000

[a] All sedimentation experiments performed at 53,220 r. p. m.
[b] See text.
[c] For anions.

ultracentrifuge used in these studies has been briefly described.[2] It employs a cylindrical lens schlieren optical system. In all experiments, the schlieren patterns were symmetrical, and values of s were readily determined from the position of the maximum ordinate of the peak as a function of time. D was calculated from the same patterns by the area-maximum ordinate method.[3] In figure 1 is plotted $(A/H_m)^2$ against t for a series of

different exposures in a representative experiment, where A is the area in cm.,[2] H_m is the maximum ordinate in cm. of the sedimenting peak, and t is the time in seconds after full rotor speed was attained. A straight line passing through the origin is obtained, which is one important criterion for a satisfactory diffusion experiment.[4] Values of $\bar{V}$ were calculated by the method of Kraemer[5] from density measurements on solutions of the crystalline hydrated compounds. The molecular weights which we have obtained should therefore best be compared with the molecular weights expected for the hydrated anions, assuming that the water of hydration in the crystal remains largely bound to the sedimenting anions. While there is some uncertainty about $\bar{V}$, as much as a 10% error in $\bar{V}$ results in an error of only 3% in the calculated molecular weight for these compounds.

Three compounds were studied: compound *a*, 12-tungsto-silicic acid, $H_4SiW_{12}O_{40}\cdot 24H_2O$, whose structure has been determined by x-ray diffraction;[6, 7] compound *b*, reported to be ammonium 11-tungsto-manganate, $(NH_4)_5H_2[Mn_2W_{22}O_{74}]\cdot 48H_2O$;[8] and compound *c*, which has been described by Rogers[9] as $13(NH_4)_2O\cdot 2P_2O_5\cdot 8V_2O_5\cdot 34WO_3\cdot 86H_2O$. Analyses indicate that this substance could probably be better represented as $(NH_4)_7[PV_4W_8O_{40}]\cdot 22H_2O$, as suggested by Dr. Pauling, and be regarded as a member of the 12-series of heteropoly compounds in which one third of the tungsten atoms have been replaced by vanadium. In table 1 are summarized the results obtained with these compounds. Within the experimental errors (largely in the diffusion constants), the calculated molecular weight of the hydrated anion $[SiW_{12}O_{40}]^{-4}\cdot 24H_2O$ agrees with the formula weight. In the case of compound *b*, the molecular weight found indicates that the probable molecular weight is one half of its formula weight. For compound *c*, a molecular weight of about 3000 for the hydrated anion is in agreement with the above suggested formulation, $[PV_4W_8O_{40}]^{-7}\cdot 22H_2O$.

Further studies of similar compounds are contemplated. In particular, it should be possible to obtain more accurate diffusion constants for these substances from measurements independent of the ultracentrifuge. These data, combined with appropriate sedimentation constants, would then provide a sensitive test of the adequacy of the Svedberg equation as applied to these small molecules. (The radii of these anions are about 7A.)

Furthermore, the compounds should be useful as standards for calibration of velocity ultracentrifuges, and as model systems for the study of a variety of hydrodynamic problems.

* Present address, Sterling Chemistry Laboratory, Yale University.

† Contribution No. 1698.

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ON VARIATIONS OF THE GEOMAGNETIC FIELD, FLUID MOTIONS, AND THE RATE OF THE EARTH'S ROTATION

By E. H. Vestine

Carnegie Institution of Washington, Department of Terrestrial Magnetism, Washington 15, D. C.

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Introduction.—Brouwer[1] has recently estimated the changes in the yearly rate of the earth's rotation since 1820. He found considerable systematic fluctuations over periods of time as short as about 50 years. These fluctuations were unexplained, though it was suggested that they are of geophysical origin.

The changes in angular momentum in the atmosphere, oceans, crust and mantle are either too small or take place too slowly to explain the major part of the observed non-seasonal changes in rate of rotation. These conclusions are also presented in a valuable paper by Walter Munk and Roger Revelle given at the spring meeting of the American Geophysical Union in 1952, at which time the present paper was read. It is understood their paper will appear in the *Monthly Notices* of the Royal Astronomical Society.

It is the purpose of the present paper to compare Brouwer's results with changes with time in the westward drift of fluid moving near the surface of the earth's central core, as inferred from the westward drift of the earth's eccentric dipole field of terrestrial magnetism.[2] Also discussed, as a check, are the effects of variation in the northerly drift of fluid in the core. The latter are less accurately inferred from the geomagnetic data, and would affect the position of the north pole, which as it moves gives a variation of latitude upon the spinning earth.

Motion of Geomagnetic Field.—It is of interest to derive the position of this eccentric dipole relative to the earth's surface as a function of time, supposing this dominant simple feature of the earth's field to indicate some corresponding broad scale feature of the fluid motion of the earth's central core. For instance, at the earth's surface the field of the eccentric dipole may be regarded as that due to a simple spherical electric current sheet flowing in the outer part of the core. Motion of the dipole about

the earth's axis of rotation at a fixed distance from the earth's center is then equivalent to a similar angular rotation of the current-sheet, and hence of the fluid in the core from Maxwell's equations.[3]

The eccentric dipole is, of course, merely a convenient physical or mathematical abstraction. It would represent the field very well indeed,

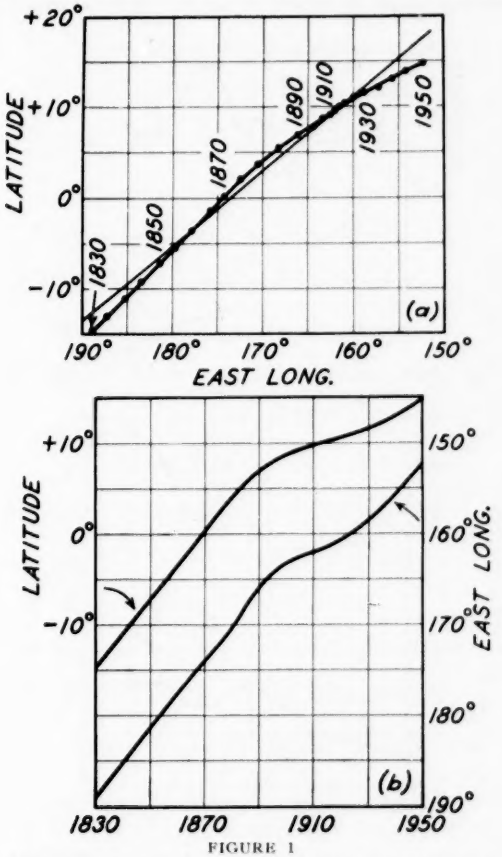

FIGURE 1

(*a*) Motion of eccentric dipole, 1830–1950, and (*b*) its latitude and longitude, 1830–1950.

however, to an observer viewing the earth from outside at a distance of a few earth radii.

This observer, rotating with the earth, would conclude that the geomagnetic field was moving westward and northward (near the East Indies), relative to the earth's surface, and also outward from the earth's center

since 1830. A fairly steady motion of the eccentric dipole westward and northward is shown in terms of latitude and longitude in figure 1. The motions have, of course, in effect been found from the migration of the eccentric dipole field, determined from magnetic measurements at the earth's surface. Since 1830 this field pattern has drifted about 35° of longitude west and about 30° north, or about 3300 km. west and 2900 km.

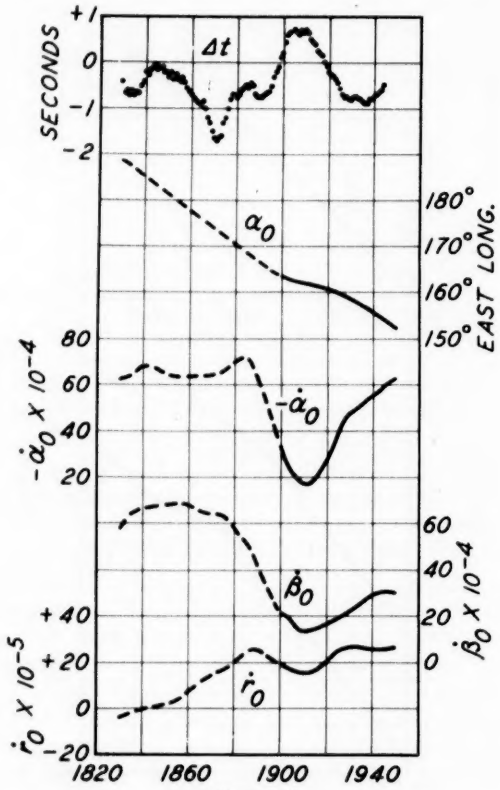

FIGURE 2

Fluctuations in core and crustal velocities, 1830–1950.

north. The equatorial plane was crossed about 1869. The motion away from the earth's center given by the change in radial distance r amounts to about 116 km. from 1830 to 1950, or at a rate of about 0.03 cm./sec. This radial motion may reflect some systematic change in asymmetry of convection within the core, which at present must remain obscure. The westward and northward motions can be understood in terms of motion of

a core layer relative to the mantle. The linear dimensions being large and the electric conductivity high necessitates that the magnetic field in the layer would move with the layer, from magnetohydrodynamics.[3] If the accompanying growth or decay of field are small, as seems likely, a rough estimate of fluid motion may be deduced.

In figure 1 a straight line has been drawn to indicate the average direc-

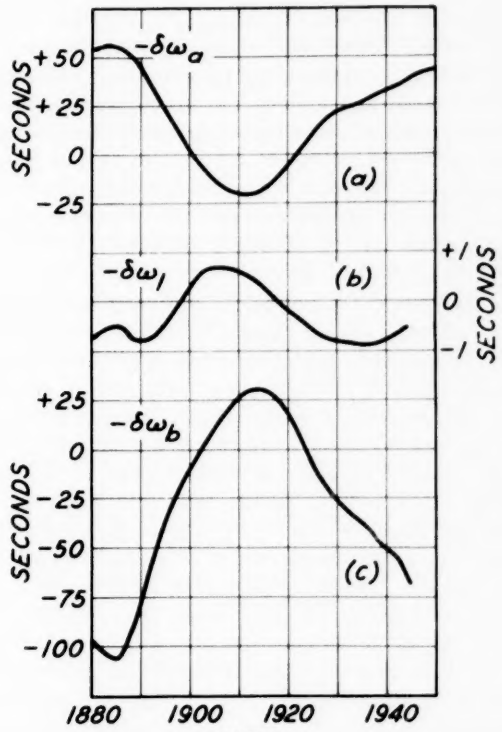

FIGURE 3

Departures in annual rate of rotation of (*a*) outer core (*b*) earth's crust, and (*c*) inner core in seconds of time, 1880–1945.

tion of motion since 1830; the north pole of this northwesterly motion (as seen from the Dutch East Indies) of the core is near Winnipeg, Canada, at 50°N latitude and 264°E longitude.

The average value of westward drift of 0°.30/year, from figure 1, compares favorably with that found by Bullard,[4] who obtained the value 0°.180 ± 0°.015/year for the average drift between the two times 1907

and 1945, using the same data in charted form. The northward drift since 1830 appears to average about 0°.25/year.

Major interest centers about the indications of fluctuations in these drift velocities, also shown in figure 1, especially near 1910. The fluctuations in yearly rates of drift are shown in figure 2 where $-\dot{\alpha}_0$ and $\dot{\beta}_0$ are the rates of westerly and "northerly" drifts in radians per annum, compared with Δt of Brouwer. These are of great importance in relation to other problems of the earth because they may imply experimentally determined changes in the energy, motion and angular momentum of the core.

The assessment of the error in the results of figure 2 is discussed in a general way in a paper to appear in *The Journal of Geophysical Research*. The conclusion reached was that the indications of fluctuation in the rates of drift of the surface magnetic field are credible, at least from about 1910 to 1945, because they depend on the precision of determination of spherical harmonic coefficients in successive years rather than upon the absolute error in the coefficients. It is highly desirable, however, that these results should be checked by future observations.

Effect of Fluctuations in Westward Drift on the Earth's Rate of Rotation.—In figure 3 results for the fluctuation in the earth's rate of rotation and the geomagnetic field in terms of time-equivalent in seconds/annum are compared. Considerable similarity in general form of the two curves is in evidence, and the sense is in the proper direction for conservation of angular momentum. A steady westward component of the magnetic drift of 4.3 sec./annum, obtained by Bullard, has been removed from the magnetic data. It turns out that the two curves have their zero values at approximately the same times. It may be noted that core motions are given relative to the mantle, but it seems convenient to neglect the small correction for the motion of the mantle itself.

The departures in angular velocity, shown in (*a*) of figure 3, necessarily apply to the outer part of the core, probably quite near the surface. This follows since a magnetic pulse of about 60 years' duration or period would otherwise fail to penetrate to the earth's surface through the highly conducting surface layers of the core, and the electrically conducting mantle.[3]

The variation of this angular velocity with depth within the core is unknown. The experimental results given in (*a*) and (*b*) of figure 3, however, have been used to determine the fluctuation in motion of the inner half by volume of the core, as in Bullard's model, in order that the total angular momentum be conserved. In this model the outer and inner halves of the core can move separately as rigid bodies.

In order that angular momentum be conserved it is necessary that the following relation be satisfied:

$$I_1\omega_1 + I_a\omega_a + I_b\omega_b = 0 \qquad (1)$$

where I_1, I_a and I_b are the moments of inertia of mantle, outer and inner core, respectively, and ω_1, ω_a and ω_b the respective angular velocities. A change $\delta\omega_1$ in ω_1 then implies

$$I_1\delta\omega_1 + I_a\delta\omega_a + I_b\delta\omega_b = 0 \qquad (2)$$

Since $\delta\omega_1$ and $\delta\omega_a$ are given by the experimental data

$$\delta\omega_b = -(I_1/I_b)\delta\omega_1 - (I_a/I_b)\delta\omega_a \qquad (3)$$

If the outer and inner concentric parts of the core have equal volumes, as might be approximately the case in convective motion, $a = 1.260\ b$, taking the average density to be the same in each, an approximation which seems adequate in a rough calculation here; there then results $(I_a/I_b) = 2.176$. From $I = 8.10 \times 10^{44}$ and $I_a + I_b = 8.58 \times 10^{43}$

$$\delta\omega_b = -26.9\ \delta\omega_1 - 2.176\ \delta\omega_a \qquad (4)$$

The results based on this equation are shown in figure 3; in addition to the fluctuations $\delta\omega_b$ there is required a steady eastward drift of $\omega_b = 0.^\circ 39$/year relative to the mantle, compensating for the steady component of westward drift.

It should next be remarked that the experimental data used do not provide any information which dictates division of the core into two parts equal by volume. There are viscous and electromagnetic couples present, of which no account has been taken. In their presence angular momentum must still be conserved.

Bullard[4] has shown that the couples due to viscous forces probably can be neglected. The viscous couple between the mantle and the core is $\pi^2\omega\eta a^3/s$, where $\eta = 10^{-2}$ is taken as the viscosity and s the thickness of the boundary layer, and ω the relative angular velocity. This is small even if $s = 1$ cm., and the time constant is very great. Although this extrapolated laboratory value of η may be incorrect ω is so small that the couple is small.

The electromagnetic couples vanish, according to Bullard, when

$$\omega_1 - \omega_a = 0.5(\omega_b - \omega_a) \qquad (5)$$

From figure 3, at times near maximum (1910) and minimum (1945) of Brouwer's curve, for the left-hand side of Bullard's equation we get 21 and -36, respectively, as compared with 25 and -42 sec. for the right-hand side. Since the value 0.5 in the formula is picked as a number between 0.32 and 0.63 by Bullard, there is no glaring discrepancy. Unfortunately, the quantities involved are too coarsely defined to render much significance to this computation indicating that the electromagnetic couples were small, if the data were correct, or alternatively, that the motions were such that the electromagnetic couples tended to vanish. It is clear, however, that though the system of motions illustrated in figure

3 is based entirely on the concept of conservation of angular momentum, such that the fluctuations in rate of rotation of the earth's surface be explained from energy supplied by the core, the operation of the system requires the existence of couples. Bullard[4] has shown that the couple between core and mantle, in the case of his model, is likely to have a large time constant, using existing laboratory and other data in estimating the electric conductivities near the core boundary. His approximate result is that the time constant $t = 2D/k_1H^2_0$, where D is the density of the core, k_1 the electric conductivity of the mantle and H_0 the magnetic field. Taking $H_0 = 4$, for $t = 5$ years, $k_1 = 7.5 \times 10^{-9}$ e. m. u.; $t = 38$, $k_1 = 10^{-9}$; $t = 76$, $k_1 = 5 \times 10^{-10}$; $t = 380$, $k_1 = 10^{-10}$; $t = 760$, $k_1 = 5 \times 10^{-11}$; and for $t = 3800$, $k_1 = 10^{-11}$. From figure 3 one is led to infer that $k_1 \sim 7.5 \times 10^{-9}$ because $t = 5$ years is more nearly appropriate if the data given here are correct. Since there is some observational evidence that small sudden pulses in secular magnetic change over periods of a few years, presumably from changes at the core, actually reach the earth's surface, it might be thought that this value k_1 is too high. Such pulses might originate at the core boundary, however, and a value as high as $k_1 = 7.5 \times 10^{-9}$ may not extend very far upward into the rigid mantle. Presumably this coarse estimate may be in error by a factor of 10.

According to Elsasser $k = C\theta^2/T$, where θ is the Debye temperature, and T the absolute temperature. The above result might suggest that some previous ideas on the temperature of the core should be revised downward from the 9000°C. suggested by others.[5]

Effect of Fluctuations in Northward Drift of Fluid in the Core.—The interpretation of the northward drift of the geomagnetic field, as shown by the motion of the eccentric dipole field indicated in figures 1 and 2, is more difficult. In the first place, it can be shown that the probable error in poleward drift is larger than in the westerly drift. In the steady westerly drift, at least, the lines of force of the dipole field are presumably wound up into two toroidal fields inside the core, opposed in direction in the northern and southern hemispheres. The motion itself is probably an expression of the influence of Coriolis force on convective motions in the core.

The apparent motion of figure 1 is that about a pole located near Winnipeg, Canada, at $\varphi = 50°$N, $\lambda = 264°$E, so that in addition to the westward component of motion there is also a motion about an axis transverse to the axis of rotation and intersecting the meridian $\lambda = 264°$E. The motion or drift of the eccentric dipole field is northward at the surface of the core near the Dutch East Indies.

The rate of "northerly" drift $\dot{\beta}_0$ in radians per year is shown in figure 2, as deduced from the motion of the eccentric dipole. The probable error in individual years is about three times that for $-\dot{\alpha}_0$. The validity

of this result is hence immediately challenged. For instance, from figure 1, the motion implied could consist of a uniform motion northward and a fluctuating motion westward. However, since the average drift velocity in figure 1 is roughly to the northwest, it is probably safe to infer that the result for $\dot{\beta}_0$ is at least qualitatively correct, and perhaps correct in order of magnitude.

The pulse in angular momentum applied transverse to the earth's axis of rotation will shift the position of the north pole relative to the axis of rotation. A variation of latitude with time will then be noted at astronomical stations observing the stars. The solution of the complex dynamical problem involved will not be attempted here, but it is of interest to show that the addition of the fluctuating angular momentum vector transverse to the axis of rotation can produce a perceptible variation in latitude. The detailed relations between the fluid motion and field may affect these results somewhat, but will not be considered here.

If the axis of rotational drift of the outer and inner core be that for the motion of figure 1, the motion about this axis has two components. The first component is that already treated to provide an explanation of fluctuations in the earth's rate of rotation, namely, the varying drifts westerly and easterly in the outer and inner core. The plane of the motion of figure 1 taken to intersect the earth's center is then inclined to the equatorial plane at about 40°. Hence the second component of motion, the varying "northerly" drift, should be given by the westerly drift multiplied by tan 40°.

The concept of a rotation of the core about an axis intersecting the earth's surface near Winnipeg, Canada, need not be particularly disturbing when it is recalled that this represents only a transient, very small in energy, when compared with that of the rapid rotation of the core and mantle about the geographic axis.

The so-called northerly drift of fluid in the outer core, as noted near the latitude and longitude of the eccentric dipole, hence applies a fluctuating pulse in angular momentum about an axis perpendicular to the axis of rotation of the spinning earth—a result of convective processes. The response of the entire system to pulses of this kind is unknown, but if it is as simple as apparently indicated by the experimental data of the westerly drift, an order of magnitude calculation can be made.

The applied angular momentum per year, represented by a vector in the equatorial plane, coincident with the axis of rotation for northerly drift, then gives

$$(I_a\delta\omega_a + I_b\delta\omega_b) \tan 40^\circ + I\omega\delta\theta = 0 \qquad (6)$$

under the condition of conservation of angular momentum. Here I_a, I_b, and I are the moments of inertia of the outer and inner core, and of the

earth, respectively, with $\delta\omega_a$, $\delta\omega_b$, the change in the westerly angular motion in radians per year and θ the angular displacement per year of the north pole, in the direction opposite to the applied vector momentum.

From figure 3, ω_a varies nearly linearly with time from 1916 to 1940 and the average is about 13 sec./year, and for ω_b about -12 sec.; on multiplication by $2\pi/86400$ these are converted to radians/year. Or using (6)

$$(-13I_a + 12I_b) \tan 40^\circ + 365 \times 86{,}400I\,\delta\theta = 0 \qquad (7)$$

which for 24 years gives $24\,\delta\theta \tan 40^\circ = 10$ feet $\times \tan 40^\circ$ toward and along the meridian including Winnipeg, at $\lambda = 264^\circ$E. This figure agrees with a recent result from astronomy.[6] If geomagnetic data since 1885 are used, the result is 10 feet $\times \tan 28^\circ = 5$ feet along $\lambda = 270^\circ$E; the axis of transient rotation of the core then intersects the earth's surface at (62°N, 270°E). It may be argued that this location of the axis is preferred, since it is based on superior data.

Conclusion.—Fluctuating fluid motions in the earth's central core are deduced from geomagnetic data. It is shown that since 1900 these motions can account qualitatively for the observed fluctuations in the earth's rate of rotation found by Brouwer. If Bullard's model of the core is used, which attributes these motions to convection, Brouwer's results are roughly in good quantitative agreement as well.

The same fluid motions also predict a variation of latitude. The amount and direction of this variation of latitude is roughly compatible with preliminary results obtained from astronomy.

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ON THE QUOTIENT VARIETY OF AN ABELIAN VARIETY

BY WEI-LIANG CHOW

THE JOHNS HOPKINS UNIVERSITY, BALTIMORE, MD.

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Let A and B be two Abelian varieties and let F be a homomorphism of A onto B; then the kernel of the homomorphism F is a subgroup in A which consists of a bunch of subvarieties in A. The question naturally arises whether conversely for any given such subgroup X in A there exist an Abelian variety B and a homomorphism of A onto B with X as its kernel. In other words, the question is whether the quotient group A/X of A over X can also be represented by an Abelian variety $A(X)$, such that the canonical homomorphism of the group A onto A/X is represented by a homomorphism of the Abelian variety A onto $A(X)$. It is the purpose of this note to show that such a quotient variety $A(X)$ always exists.

We shall begin with a few definitions.[1] A subgroup X in an Abelian variety A is said to be an *algebraic subgroup* if X is a bunch of subvarieties in A, i.e., if X is the union of a finite number of subvarieties in A. An algebraic subgroup X is said to be *algebraic* or *normally algebraic* over a field K if as a bunch of subvarieties it is algebraic or normally algebraic over K; X is said to be *separable* over K, if each component variety in the bunch X is defined over a separable algebraic extension of K. It is known[2] that all component varieties of an algebraic subgroup X in A are disjoint to one another, and that the one and only component variety X_0 containing the unit element is an Abelian variety and the other component varieties are obtained from X_0 by translations with respect to some elements in A. The dimension of X_0, which is also the dimension of all other component varieties in X, is called the *dimension* of X. It is clear that if X is normally algebraic over K, then X_0 is defined over a purely inseparable extension of K, and if, furthermore, X is also separable over K, then X_0 is defined over K.

We recall here the fact[3] that the aggregate of all positive cycles of a given dimension r and a given degree d (i.e., the degree in the ambient projective space) in an algebraic variety V can be represented in a one-to-one and algebraic manner by the points of a bunch of varieties, normally algebraic over K, which we shall denote by $M(V; r, d)$. We shall call this representation the *canonical mapping* Γ of the positive r-cycles into $M(V; r, d)$; this mapping has the property that if a positive r-cycle Z of degree d in V is rational over a field K', then the point $\Gamma(Z)$ in $M(V; r, d)$ is also rational over K'.

Let F be a rational transformation of an algebraic variety V and let K be a field of definition for both V and F, then F is said to be *separable* if $K(x)$ is a separably generated extension of $K(F(x))$, where x is a generic

point of V over K. Since $K(x)$ is separably generated over K, this definition of separability is independent of the choice of the field K.

Let F be a homomorphism of an Abelian variety A into an Abelian variety B, and let K be a field over which A, B, F are all defined. Then the kernel of F, being the projection in A of the intersection of $A \times (0)$ with graph of F in the product variety $A \times B$, is an algebraic subgroup in A, normally algebraic over K; if furthermore the homomorphism F is separable, then this algebraic subgroup is also separable over K.

THEOREM 1. *Let A be an Abelian variety of dimension n and let X be an algebraic subgroup of dimension r in A. Then the quotient group A/X can be represented by an Abelian variety $A(X)$ of dimension $n-r$, such that the canonical homomorphism of the group A onto A/X is represented by a separable homomorphism F of the Abelian variety A onto $A(X)$. Furthermore, if H is any homomorphism of A into an Abelian variety B with X as the kernel, then H is the product of F and a one-to-one homomorphism of $A(X)$ into B. If K is a field such that A is defined over K and X is normally algebraic and separable over K, then both $A(X)$ and F are defined over K.*

Proof: Since X is a bunch of varieties of dimension r, normally algebraic and separable over K, it determines a rational positive r-cycle Z over K in A, which contains each variety in X as a simple component. Let x be a generic point of A over K, and let T_x be the bi-regular birational transformation of A onto itself given by the translation attached to x; then $T_x(Z)$ is a rational positive r-cycle over $K(x)$ in A, which of course also contains each variety in $T_x(X)$ as a simple component. If d is the degree of $T_x(Z)$ as a positive r-cycle in the ambient projective space of A, then the canonical mapping Γ of the positive r-cycles of degree d in A onto $M(A; r, d)$ induces a rational transformation $F(x) = \Gamma(T_x(Z))$ of A into $M(A; r, d)$. It is clear that the image $F(A)$ of A is a variety $A(X)$ defined over K; we shall show that $A(X)$ has the properties stated in our theorem.

Let W be the graph of the law of composition in A; it is known[4] that for any point x_0 in A, we have $x_0 \times T_{x_0}(Z) = pr_{13}[W \cdot (x_0 \times Z \times A)]$. This shows at once the $T_x(Z)$ has the unique specialization $T_{x_0}(Z)$ over the specialization $x \rightarrow x_0$ for any point x_0 in A. It follows then that $F(x)$ has a unique specialization $F(x_0) = \Gamma(T_{x_0}(Z))$ over every specialization $x \rightarrow x_0$, and since A is non-singular, this implies that the rational transformation $F(x)$ is defined everywhere in A. If x_1 is another point in A, then we have evidently $F(x_0) = F(x_1)$ if and only if $T_{x_0}(Z) = T_{x_1}(Z)$. In particular, for the generic point $y = F(x)$ in $A(X)$ over K, the inverse image $F^{-1}(y)$ is a prime rational cycle over $K(y)$ which contains the same component varieties as the cycle $T_x(Z)$; and since $T_x(Z)$ is rational over $K(y)$ and has only simple components, it follows that $F^{-1}(y) = T_x(Z)$ and hence also $F^{-1}(y_0) = T_{x_0}(Z)$ for every x_0 and $y_0 = F(x_0)$. According to a result of ours proved elsewhere,[5] all this implies that the variety

$A(X)$ is a non-singular variety; furthermore, it is clear that the rational transformation F is separable.

Next we shall show that $A(X)$ is an Abelian variety. Let y_1 and y_2 be two independent generic points of $A(X)$ over K, and let x_1 and x_2 be generic points of the prime rational cycles $F^{-1}(y_1)$ and $F^{-1}(y_2)$ over $K(y_1, y_2)$, respectively, independent with respect to each over $K(y_1, y_2)$. The point $F(x_1 + x_2)$ is then rational over $K(y_1, y_2, x_1, x_2)$. Let x_1' and x_2' be generic points of $F^{-1}(y_1)$ and $F^{-1}(y_2)$, respectively, over $K(y_1, y_2)$, independent with respect to each other as well as to the points x_1 and x_2 over $K(y_1, y_2)$; then the point $F(x_1' + x_2')$ is rational over $K(y_1, y_2, x_1', x_2')$. Since x_1' is in $T_{x_1}(Z)$ and x_2' is in $T_{x_2}(Z)$, the point $x_1' + x_2'$ is in $T_{x_1+x_2}(Z)$; hence we have $F(x_1 + x_2) = F(x_1' + x_2')$. This shows that the point $F(x_1 + x_2)$ is rational over $K(y_1, y_2, x_1, x_2) \cap K(y_1, y_2, x_1', x_2') \subset K(y_1, y_2)$ and is independent of the choice of the generic points x_1 and x_2. Since x_1 and x_2 are separably generated over $K(y_1, y_2)$, the point $F(x_1 + x_2)$ is also separably generated over $K(y_1, y_2)$; and since $F(x_1 + x_2)$ is uniquely determined by the points y_1 and y_2, it follows that $F(x_1 + x_2)$ is rational over $K(y_1, y_2)$. We shall now define $y_1 + y_2$ to be the point $F(x_1 + x_2)$ and show that this defines a law of composition in $A(X)$, defined everywhere in $A(X)$. The group properties of this law of composition follow easily from the same properties in A, and can be indicated as follows: $y_1 + (y_2 + y_3) = F(x_1 + (x_2 + x_3)) = F((x_1 + x_2) + x_3) = (y_1 + y_2) + y_3$, $-y = F(-x)$, and the unit element in $A(X)$ is evidently $F(O)$. It only remains to show that the function $y_1 + y_2$ is defined everywhere in the product variety $A(X) \times A(X)$; since $A(X)$ is non-singular, we only need to show that $y_1 + y_2$ has a unique specialization over every specialization $y_1 \to \eta_1$, $y_2 \to \eta_2$. Now, every such specialization $y_1 + y_2 \to \eta_3$ can be extended to a specialization of $y_1 + y_2 \to \eta_3$, $x_1 \to \xi_1$, $x_2 \to \xi_2$ over the specialization $y_1 \to \eta_1$, $y_2 \to \eta_2$, and in this specialization the relations $y_1 = F(x_1)$, $y_2 = F(x_2)$, $y_1 + y_2 = F(x_1 + x_2)$ go over into the relations $\eta_1 = F(\xi_1)$, $\eta_2 = F(\xi_2)$, $\eta_3 = F(\xi_1 + \xi_2)$. Since $F(\xi_1 + \xi_2)$ remains fixed for every choice of the points ξ_1 and ξ_2 in $F^{-1}(\eta_1)$ and $F^{-1}(\eta_2)$ respectively, the point η_3 is therefore uniquely determined by η_1 and η_2.

Consider now the homomorphism H of A into B; let K' be a field over which A, B, $A(X)$, F and H are defined, and let x be a generic point of A over K'. Then $H(A)$ is an Abelian subvariety in B, defined over K', and the point $z = H(x)$ is a generic point of $H(A)$ over K'. Since F is separable and is defined over K', $K'(x)$ is a separably generated extension of $K'(y)$, and hence $K'(y, z)$ is also a separably generated extension of $K'(y)$. Let y_0 be a point in $A(X)$, and let $x \to x_0$, $z \to z_0$ be specializations over the specialization $y \to y_0$ over K'; then the relations $F(x) = y$, $H(x) = z$ go over into the relations $F(x_0) = y_0$, $H(x_0) = z_0$. The fact that H has X as its kernel implies that $H(x_0) = z_0$ is a uniquely determined

point for all x_0 satisfying the relation $F(x_0) = y_0$, so that z has a unique specialization over every specialization of y over K'. This implies not only that z is algebraic over $K'(y)$ but also that z is rational over a purely inseparable extension of $K'(y)$; and since $K'(y, z)$ is a separably generated extension of $K'(y)$, it follows that z is rational over $K'(y)$. The rational function $z = H_1(y)$ then defines a rational transformation of $A(X)$ into B, defined over K', which can be easily seen to be a homomorphism, and we have evidently the equation $H(x) = H_1F(x)$. This concludes the proof of our theorem.

Theorem 1 gives a satisfactory answer to our question from the point of view of "absolute" algebraic geometry. However, one is often interested in the properties of a variety with respect to a given field of definition, and the consideration of such relative properties can be significant even in cases where one's final objective is "absolute." From the relative point of view it is quite natural to take as K a field of definition of A over which the algebraic subgroup X is normally algebraic, as otherwise a quotient variety with the specified properties cannot possibly be defined over K; but the additional condition that X be separable over K seems to be an undesirable restriction. It is therefore not without some interest to show (Theorem 2) that there exists a relative quotient variety $A(X, K)$ over any field of definition K of A over which X is normally algebraic; this relative quotient variety $A(X, K)$ has similar properties as $A(X)$, though naturally in a somewhat restricted form.

We shall begin with two simple lemmas and a few definitions.

LEMMA 1. *Let K' be a finitely generated extension of K, and let $\mathfrak{V}$ be an aggregate of subfields in K' which all contain K. If $\mathfrak{V}$ has the property that the compositum of every two fields in $\mathfrak{V}$ is again a field in $\mathfrak{V}$, then the subfield in K' generated by all fields in $\mathfrak{V}$ is also a field in $\mathfrak{V}$.*

Proof: Let L_1 be a field in $\mathfrak{V}$ which has the maximum possible dimension over K; then for any field L in $\mathfrak{V}$, the compositum LL_1, which by assumption is in $\mathfrak{V}$, is a finite algebraic extension of L_1. Since $K' \cap \bar{L}_1$ is a finite algebraic extension of L_1 and since LL_1 is contained in $K' \cap \bar{L}_1$, the degree of LL_1 over L_1 is bounded for all L in $\mathfrak{V}$. Let L_2 be a field in $\mathfrak{V}$ such that L_2 contains L_1 and has the maximum possible degree over L_1; then for every field L in $\mathfrak{V}$, the fields LL_2 and L_2 must have the same degree over L_1 and hence must coincide. This shows that L_2 is the subfield in K' generated by all the fields in $\mathfrak{V}$.

If an aggregate $\mathfrak{V}$ of subfields in K' has the property stated the preceding lemma, then the subfield in K' generated by all the subfields in $\mathfrak{V}$ is called the *maximal field* of $\mathfrak{V}$.

A field L containing K is said to be an *Abelian field over K*, if it is the function field of an Abelian variety defined over K. Such a field L is of course a finitely generated extension, and the Abelian variety A associated

with it is uniquely determined up to an isomorphism over K. If L_1 is an Abelian subfield over K in L and if A_1 is the Abelian variety associated with L_1, then there is associated with L_1 a homomorphism F of A onto A_1, defined over K, which is determined up to isomorphisms of A and A_1.

LEMMA 2. *Let L be an Abelian field over K, and let L_1 and L_2 be two Abelian subfields over K in L. Then the compositum L_1L_2 is also an Abelian field over K; furthermore, if L_1 is associated with a homomorphism of A with the kernel X_1 and L_2 is associated with a homomorphism of A with the kernel X_2, then L_1L_2 is associated with a homomorphism of A with the kernel $X_1 \cap X_2$.*

Proof: Let A, A_1, A_2 be the Abelian varieties associated with L, L_1, L_2, respectively, and let F_1 and F_2 be the associated homomorphisms of A onto A_1 and A_2, respectively, with X_1 and X_2 as the kernels, respectively. Then the rational transformation $F_1 \times F_2$ is a homomorphism of A into the product variety $A_1 \times A_2$, and the image of A under this homomorphism is then an Abelian variety A_3 over K whose function field is isomorphic to L_1L_2. It is clear that the homomorphism $F_1 \times F_2$ of A onto A_3 has the kernel $X_1 \cap X_2$.

THEOREM 2. *Let A be an Abelian variety of dimension n, defined over K, and let X be an algebraic subgroup of dimension r in A, normally algebraic over K. Then the quotient group A/X can be represented by an Abelian variety $A(X, K)$, defined over K, such that the canonical homomorphism of the group A onto A/X is represented by a homomorphism F of the Abelian variety A onto $A(X, K)$, also defined over K; furthermore, if H is any homomorphism of A into an Abelian variety B with X as the kernel and if both B and H are defined over K, then H is the product of F and a one-to-one homomorphism of $A(X, K)$ into B, also defined over K.*

Proof: Since X is normally algebraic over K, there exists a positive integer λ such that X is separable over K^{-p^λ}; then according to Theorem 1 the Abelian variety $A(X)$ is defined over the field K^{-p^λ}. Let y be a generic point of $A(X)$ over K^{-p^λ}, and consider the field $K(y^{p^\lambda})$, where y^{p^λ} denotes the point obtained from y by raising all coordinates of y to the p^λth power. Since $K^{-p^\lambda}(y)$ and $K(y^{p^\lambda})$ are isomorphic under mapping of raising each element of the former to its p^λth power, and since $K^{-p^\lambda}(y)$ is an Abelian field of dimension n-r over K^{-p^λ}, it follows that $K(y^{p^\lambda})$ is an Abelian field of dimension n-r over K. Also from $K^{-p^\lambda}(x) \supset K^{-p^\lambda}(y)$ follows $K(x^{p^\lambda}) \supset K(y^{p^\lambda})$ and hence $K(x) \supset K(y^{p^\lambda})$.

Consider now the aggregate $\mathfrak{V}$ of all Abelian subfields over K in $K(x)$ with the property that each subfield has an associated homomorphism with X as its kernel; this aggregate $\mathfrak{V}$ is not empty as $K(y^{p^\lambda})$ is one such subfield. Furthermore, it is easily seen that the compositum of two fields in $\mathfrak{V}$ is also a field in $\mathfrak{V}$; in fact, according to Lemma 2 this compositum is also an Abelian subfield over K and it has an associated homomorphism

with $X \cap X = X$ as its kernel. It follows then from Lemma 1 that $\mathfrak{Y}$ has a maximal field L, and we can define $A(X, K)$ to be the Abelian variety determined by L and define F to be the corresponding homomorphism of A onto $A(X, K)$. It is clear that the so-defined $A(X, K)$ and F have the properties stated in our theorem.

[1] We shall assume that the reader is familiar with the contents of the book by A. Weil, *Variétés abéliennes et courbes algébriques*, Paris, 1948, and we shall use the terminology of Weil as developed in this book, with the exception that all our varieties are understood to be embedded in some projective spaces. Since a point x in a projective space is given by a system of homogeneous coordinates, we shall denote by $K(x)$ the field obtained from K by the adjunction of all the ratios (with non-zero denominators) of the coordinates of x.

[2] Weil, *loc. cit.*, No. 24, Proposition 8, p. 39.

[3] Chow, W. L., and van der Waerden, B. L., "Zur algebraischen Geometrie. IX. Ueber zugeordnete Formen und algebraische Systeme von algebraischen Mannigfaltigkeiten," *Math. Ann.*, Band 113, 692–704 (1937).

[4] Weil, *loc. cit.*, No. 13, Théorème 4, p. 22.

[5] Chow, W. L., "Algebraic Systems of Positive Cycles in an Algebraic Variety," *Am. J. Math.*, **72**, 247–283 (1950).

*A THEOREM ABOUT CHARACTERISTICS OF DIFFERENTIAL EQUATIONS ON CLOSED MANIFOLDS**

By Felix Haas

Department of Mathematics, Lehigh University, Bethlehem, Pa.

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The following is concerned with the in-the-large behavior of characteristics of differential equations on closed orientable manifolds. C^+ will denote a positive semicharacteristic, $\bar{C}$ the set of ω-limit points of C^+, V the vector field defining the differential equation, M the manifold under consideration and W the vector field orthogonal to V. The following theorem holds:

Theorem. *If the vector field V satisfies a Lipschitz condition and has at most a denumerable number of singular points, and if $\bar{C}$ for a certain C^+ does not contain any singular points then either M is a torus and V is free of singular points or $\bar{C}$ is nowhere dense on M if M is two-dimensional.*

Remark: If M is a torus and V is free of singular points then the work of Denjoy[1] shows that $\bar{C}$ may be dense on M.

The following is a sketch of the proof of the above theorem. A more detailed proof will appear shortly.

Definition: $\bar{C}$ belongs to Class II if and only if an infinite W-characteristic or a closed W-characteristic belongs to $\bar{C}$. Otherwise $\bar{C}$ belongs to Class I.

LEMMA 1. *AB is an arc of length σ which is part of the V-characteristic C_A^+. This V-characteristic remains at a finite distance from the singular points of V. D_A^+ and D_A^- are the orthogonal semicharacteristics through A. The arc AB is not a closed curve. Then there exists an $\epsilon > 0$ and a map ϕ of the rectangle $\{(\rho, \tau) | -\epsilon < \rho < \epsilon, 0 \leq \tau \leq \sigma\}$ into M such that (1) ϕ is a homeomorphism; (2) $\phi(0, \tau)$ is the point on the arc AB at an arc length distance τ from A; (3) for $\rho > 0$, $\phi(\rho, 0)$ is the point on D_A^+ at an arc length distance ρ from A and for $\rho < 0$ the same statement holds with D_A^- taking the place of D_A^+; (4) for $\rho > 0$, $\phi(\rho, \tau)$ belongs to the intersection of the positive V-characteristic through $\phi(\rho, 0)$ with the positive W-characteristic through $\phi(0, \tau)$ and for $\rho < 0$ the same statement holds with the negative W-characteristic taking the place of the positive W-characteristic.*

Lemma 1 follows from the fundamental existence theorem and some elementary trigonometric considerations.

We first prove that if $\bar{C}$ belongs to Class I then $\bar{C}$ is nowhere dense. This proof will proceed by contradiction. Let p be an interior point of G, the set in which $\bar{C}$ is dense. Then p is an interior point of $\bar{C}$ for all of G belongs to $\bar{C}$ since $\bar{C}$ is a closed point set. Let D be the orthogonal characteristic through p, E the intersection of $\bar{C}$ and D, and K the component of E in the one-dimensional topology of D to which p belongs. K is an interval since $\bar{C}$ belongs to Class I and is closed by the properties of $\bar{C}$.

LEMMA 2. *Assuming that $\bar{C}$ belongs to Class I and that p and K have the same meaning as above we let p^* be the first intersection of C_p^+ with K (which we may assume exists; the interior of K must contain points of C^+. Hence we may just assume that p belongs to C^+. Then the characteristic through p will return to K). ρ and τ are normalized parameters on K and the arc pp^*, respectively, with $\rho(p)$ equal to ρ_0. Then there exists a topological space X and a map ϕ of X into M such that (1) X is obtained from the unit square by identifying the edge $\tau = 1$ with a closed subset of the edge $\tau = 0$ in a sense-preserving manner; (2) ϕ is a homeomorphism; (3) $\phi(\rho, 0)$ is the point of K with parametric value ρ; (4) $\phi(\rho_0, \tau)$ is the point of the arc pp^* with parametric value τ; (5) $\phi(\rho, \tau)$ belongs to $C_\rho^+ \cap D_\tau^+$ if $\rho > \rho_0$ and to $C_\rho^+ \cap D_\tau^-$ if $\rho < \rho_0$, where C_ρ^+ is the positive V-characteristic through the point of K with parametric value ρ and D_τ^+ and D_τ^- are defined similarly in terms of the orthogonal vector field.*

Proof of Lemma 2: Let J be the maximal interval around ρ_0 in $(0 \leq \rho \leq 1)$ such that the set $\{(\rho, \tau) | \rho \in J, 0 \leq \tau \leq 1\}$ is a subset of a space X described in the statement of Lemma 2 and such that there exists a map ϕ of this set into M which satisfies conditions 1–5 of the lemma and such that $\phi(\rho, \tau)$ belongs to K if and only if $\tau = 0$ or $\tau = 1$. If we can show that J is the whole unit interval we have proved the lemma.

First, J is open by Lemma 1. Let $J_{\tau_0} = \phi\{(\rho, \tau) | \rho \in J$ and τ is fixed$\}$. Since we assumed that $\bar{C}$ belongs to Class I J_{τ_0} is a finite interval on an

orthogonal trajectory. If J is not closed let $\bar{p}$ be the end point of J which does not belong to J. Let $\Gamma(\bar{p}, \tau_0)$ be the corresponding end point of J_{τ_0}. We then apply Lemma 1 to the arc $\phi(p_0, \tau_0)\Gamma(\bar{p}, \tau_0)$ on the orthogonal trajectory. Then for every τ_0 there exists a number $\epsilon(\tau_0)$ and a local map Ψ_{τ_0} described in Lemma 1. One then shows that Ψ_{τ_0} and ϕ coincide for the intersection of domains. Hence, $\Psi_{\tau_0}(\bar{p}, \tau) = \Gamma(\bar{p}, \tau)$ if $|\tau - \tau_0| < \epsilon(\tau_0)$. A finite number of the $\epsilon(\tau_0)$ cover the range $0 \leq \tau \leq 1$ by compactness. One thus finds that $\Gamma(\bar{p}, \tau)$ is on $C_{\bar{p}}^+$ for all τ and that the map ϕ can be extended to $\{(\bar{p}, \tau)|0 \leq \tau \leq 1\}$ by just defining $\phi(\bar{p}, \tau) = \Gamma(\bar{p}, \tau)$. This would imply that J is not maximal, which in turn proves that J is closed.

LEMMA 3. *If $\bar{C}$ is dense in some set G of M then $\bar{C}$ cannot belong to Class* I.

Proof of Lemma 3: Let p, p^*, D and K have the same meaning as before. By Lemma 2 all the V-characteristics starting at K return to K and the map $*$, which is defined for all points of K by q^* is the first intersection of C_q with K, is a sense-preserving homeomorphism. (q is an arbitrary point of K.) It has previously been remarked that we may assume that p belongs to C^+. Hence p^* cannot equal p; for if it did then C^+ would be a closed curve and all of K would not belong to $\bar{C}$. On the other hand, if $p \neq p^*$, in view of the sense-preserving nature of the transformation $*$, successive intersections of $C_p{}^+$ would all intersect K on the same side of p. This contradicts the fact that there are ω-limit points of C^+ on the other side of p.

LEMMA 4. *If $\bar{C}$ belongs to Class* II *then there exists a closed curve S with continuous curvature such that the solution curves of V are never tangent to S and always cross S in the same direction and such that S belongs to $\bar{C}$.*

Proof of Lemma 4: This is proved very much like a similar lemma of Siegel's.[2] It is obviously true if a closed curve characteristic of the orthogonal vector field belongs to $\bar{C}$. Otherwise we just connect two points b_1 and b_2 on the infinite orthogonal characteristic which belongs to $\bar{C}$ by a line segment. The line segment b_1b_2 together with the arc b_1b_2 on the orthogonal characteristic form the desired closed curve.

LEMMA 5. *If $\bar{C}$ is dense in some set of M then $\bar{C}$ cannot belong to Class* II *unless M is a torus and V is free of singular points.*

Proof of Lemma 5: In the proof of this lemma we make use of the closed curve constructed in Lemma 4. We define parallels to this closed curve. These parallels are generally just orthogonal trajectories, but they become straight lines in the neighborhood of the straight line part of S. We then prove that all V-characteristics starting on S return to S, and that, in fact, a torus obtained from the unit square by double identification can be mapped homeomorphically onto the point set generated by these characteristics.

The proof of this statement follows very closely the proof of Lemma 2. The major difference is due to the fact that in the proof of Lemma 2 in the part where J was proved to be closed we made use of the fact that $\bar{C}$ belongs to Class I and that hence J_{τ_0} is a finite interval. We must now show that if J is not the whole unit interval then J_{τ_0} cannot be of infinite arc length.

We prove this fact by contradiction. Suppose that $J\tau_0$ is of infinite arc length and that J is not the whole unit interval. This means in particular that the image under ϕ of the set $\{(\rho, 0)|\rho \epsilon J\}$ is not all of S. We can construct a closed curve T from $J\tau_0$ and an arc of arbitrarily short length just as in Lemma 4. Let c_1 and c_2 be the points on $J\tau_0$ which are connected by a line segment. (We may assume that the line segment chosen makes an angle of between 40° and 50° with the vectors of V and of W at c_1 and c_2). Then $C_{c_1}{}^+$ or $C_{c_1}{}^-$ will intersect the W-characteristic through c_2 at a point c_3, and the arc length of the arc c_1c_3 on the V-characteristic will be less than four times the length of the segment c_1c_2. If we use ϕ for the map from the torus or its subset determined by J into M then $c_1 = \phi(\rho_1, \tau_0)$ and $c_3 = \phi(\rho_2, \tau_0)$ for some ρ_1 and ρ_2. The nature of ϕ is such that if c_3 belongs to $C_{c_1}{}^+$ then $\phi(\rho_2, 0)$ is between $\phi(\rho_1, \tau_0)$ and $\phi(\rho_2, \tau_0)$, and if c_3 belongs to $C_{c_1}{}^-$ then $\phi(\rho_1, 0)$ is between $\phi(\rho_2, \tau_0)$ and $\phi(\rho_1, \tau_0)$. In either case we can show that for every $\epsilon > 0$ there exists $\rho(\epsilon)$ belonging to J such that the arc $\phi(\rho(\epsilon), 0)\phi(\rho(\epsilon), \tau_0)$ is shorter than ϵ. On the other hand, if we use the fact that there can be no infinite parallel to S near S we can show that there exists $\epsilon_0 > 0$ such that the arcs $\phi(\rho_0, 0)\phi(\rho, \tau_0)$ are longer than ϵ_0 for all ρ belonging to J. Hence $J\tau_0$ cannot be of infinite arc length. Once we have proved that a torus can be mapped into M in the indicated way Lemma 5 is proved.

Lemmas 3 and 5 prove the theorem.

* This work was supported by the Office of Naval Research.

[1] Denjoy, A., *J. Math. Pures Appl.* (9), **11**, 333–375 (1932).

[2] Siegel, C. L., "Note on Differential Equations on the Torus," *Ann. Math.*, **46** (3), 423–428 (1945).

ON THE COMPLETENESS OF QUANTIFICATION THEORY

By Burton Dreben

Society of Fellows, Harvard University

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In 1930 Herbrand showed that any quantificational schema in prenex normal form is demonstrable if and only if a truth-functional tautology possessing certain properties is constructible from the schema. He further showed that if any such schema is demonstrable, it is demonstrable by certain elementary rules (see Theorem I) without the help of *modus ponens*.[1] In the same year, Gödel proved that every valid quantifica-

tional schema is demonstrable.[2] By combining parts of Herbrand's reasoning with Gödel's, we may easily obtain a somewhat new argument for Gödel's Completeness Proof in which no recourse is had to Skolem Normal Forms and in which the dispensability of *modus ponens* is made clear. Such an argument, which incidentally seems to afford a rather simple method for discovering the provability of quantificational schema, is briefly presented in this paper. Some observations also will be made about the relation between Gödel's and Herbrand's results, and about the distinction drawn by Hilbert and Bernays between finitistic (proof-theoretic) and set-theoretic investigations of quantification theory.[3]

THEOREM I. *Every quantificationally valid schema ϕ in prenex normal form is demonstrable. Every such demonstration, moreover, can be given a form, called a Herbrand Normal Form, which consists (a) of a first line which is a purely truth-functional tautology, and (b) of successive lines each of which is obtained from its immediate predecessor by applying one of the following three rules I, II, III:*

Given an alternation χ as a line of a proof:

(I) Replace any alternation clause ϕ^{δ}_{ρ} of χ by $\ulcorner(\exists\rho)\phi\urcorner$, where ρ is any arbitrary variable such that if it appears in ϕ^{δ}_{ρ} it is δ;[4]

(II) Replace any alternation clause ϕ^{δ}_{ρ} of χ by $\ulcorner(\rho)\phi\urcorner$, where δ is not free in any other clause of χ nor in ϕ and ρ is any arbitrary variable such that if it appears in ϕ^{δ}_{ρ} it is δ;

(III) Drop any repetitions of an alternation clause of χ.

Proof: Let ϕ be any quantificational schema in prenex normal form without free variables and with a scope ϕ'.

Let the ϕ-order of the variables of ϕ' be the order of the quantifiers in the prenex reading from left to right.

Let the total number of existential variables be k and be denoted in ϕ-order by $\alpha_1, \alpha_2, \ldots, \alpha_k$; and the total number of universal variables be j and be denoted in ϕ-order by $\beta_1, \beta_2, \ldots, \beta_j$.

Now consider an infinite set of distinct variables foreign to ϕ, denoted in alphabetical order by $\gamma_1, \gamma_2, \ldots$; and let γ stand for any one of them. Construct a series of ordered k-ads of these variables such that

(*a*) if the sum of the subscripts of two k-ads differ, that k-ad is earlier in the series whose sum is less,

(*b*) if the sum of the subscripts of two k-ads agree, that k-ad is earlier which is earlier in lexicographic order.

To each ith ($i = 1, 2, \ldots$) k-ad associate an expression ψ_i thus:

(1) ψ_1 is formed from ϕ' by putting for $\alpha_1, \ldots, \alpha_k$ the first k-ad and by putting for $\beta_1, \ldots, \beta j$, respectively, either $\gamma_1, \ldots, \gamma_j$ or $\gamma_2, \ldots, \gamma_{j+1}$ according as ϕ does or does not begin with a universal quantifier.

(2) Assume ψ_1 to ψ_{i-1} have been obtained. Let g ($g = 1, \ldots, k$) either be the number of the *latest* place in the ith k-ad such that there is

an hth k-ad ($h = 1, \ldots, i - 1$) each of whose variables in places 1 to $g - 1$ are the same as the variables in places 1 to $g - 1$ of the ith k-ad; or be the number one if there is no such hth k-ad, and in which case let ψ_h be ψ_1 in what follows. Then form ψ_i from ϕ' by

(a) putting for $\alpha_1, \ldots, \alpha_k$ the ith k-ad,

(b) putting for each β_c ($c = 1, \ldots, j$) (if any) which precedes in ϕ-order the gth existential variable α_g ($g = 1, \ldots, k$) the same γ_w ($w = 1, \ldots$) which was put for β_c to form ψ_h,

(c) putting for each β_f ($f = 1, \ldots, j$) (if any) which succeeds in ϕ-order α_g the alphabetically earliest γ which has occurred neither in $\psi_1, \ldots, \psi_{i-1}$ nor thus far in ψ_i nor in the ith k-ad.[5]

Now either there is some number n for which $\ulcorner \psi_1 \vee \psi_2 \vee \ldots \vee \psi_n \urcorner$ is a tautology, called a Herbrand Tautology, or there is no such n.

PART A: *If $\ulcorner \psi_1 \vee \ldots \vee \psi_n \urcorner$ is a tautology, then ϕ is deducible from it solely by use of rules I, II and III.*

Proof: Let ψ_i^m ($i = 1, \ldots, n$) ($m = 0, \ldots, k + j$) be the alternand obtained from ψ_i by m uses of rules I and II such that the m distinct quantifiers of ψ_i^m are the last m quantifiers of ϕ in the prenex order of ϕ.

Let us say that rules I and II are *appropriately applied* to some alternand ψ_i^m or *appropriately applied* to some free variable γ in ψ_i^m, if the quantifiers resulting from such application are those of ϕ in the prenex order of ϕ.

Let α_f^i ($i = 1, \ldots, n$) ($f = 1, \ldots, k$) be the γ_s ($s = 1, \ldots$) which stands in ψ_i wherever α_f stands in ϕ'; and let β_u^i ($u = 1, \ldots, j$) be the γ_d ($d = 1, \ldots$) which stands in ψ_i wherever β_u stands in ϕ'.

Let the ψ_i^m-order of the free variables in ψ_i^m be the same as the ϕ-order of the variables they supplant in ϕ'. (Note that in general the ψ_i^m-order differs from the alphabetic order of the γ's.)

Each ψ_i ($i = 1, \ldots, n$) differs from ϕ' just with respect to individual variables. Moreover, in each ψ_i, $\beta_c^i = \beta_u^i$ if and only if $\beta_c = \beta_u$, $\beta_c^i = \alpha_f^i$ only if α_f is later than β_c in the ϕ-order of ϕ', and β_u^i is alphabetically later than β_c^i if and only if β_u is later than β_c in the ϕ-order of ϕ'. Hence the sole apparent difficulty in deriving ϕ from $\ulcorner \psi_1 \vee \ldots \vee \psi_n \urcorner$ by appropriately applying rules I and II and dropping repetitions lies in the first restriction on the use of rule II. This restriction however, as we shall see, will never prevent the obtaining of ϕ, if rules I, II and III are so employed that rule II is always appropriately applied to the alphabetically latest β_c^i, and that after each appropriate application of either rules I or II all repetitious alternands are dropped by rule III.

To see that rules I, II and III when thus employed always yield ϕ from $\ulcorner \psi_1 \vee \ldots \vee \psi_n \urcorner$, we shall assume that sometimes they do not and then show this assumption false. Now ϕ is not derivable only if at some point rule II has to be appropriately applied to a variable β_c^i ($c = 1, \ldots, j$) in an alternand ψ_i^m and there is another alternand ψ_l^p ($l = 1, \ldots, n$)

($p = 0, \ldots, k + j$) containing a free variable γ such that $\gamma = \beta_c{}^i$ and γ is not removable by either rule I or rule II. So we shall assume there are two such alternands $\psi_l{}^m$ and $\psi_l{}^p$.

Being a variable in $\psi_l{}^p$, γ either is some variable $\alpha_f{}^l$ ($f = 1, \ldots, k$) or some variable $\beta_w{}^l$ ($w = 1, \ldots, j$).

Case 1: $\gamma = \alpha_f{}^l$ and hence $\beta_c{}^i = \alpha_f{}^l$. Then there are two subcases:

(*i*) There is no variable $\beta_d{}^l$ ($d = 1, \ldots, j$) which succeeds $\alpha_f{}^l$ in the $\psi_l{}^p$-order of $\psi_l{}^p$. Then by successive appropriate applications of rule I, $\alpha_f{}^l$ is removable before rule II has to be applied to $\beta_c{}^i$. So the assumption fails.

(*ii*) There is a variable $\beta_d{}^l$ which succeeds $\alpha_f{}^l$ in the $\psi_l{}^p$-order of $\psi_l{}^p$. Then by the construction of ψ_l, $\beta_d{}^l$ is alphabetically later than $\alpha_f{}^l$ or $\beta_c{}^i$. But this is impossible for, by hypothesis, $\beta_c{}^i$ is the alphabetically latest free variable supplanting a universal variable of ϕ. Hence the assumption fails.

Case 2: $\gamma = \beta_w{}^l$, and hence $\beta_c{}^i = \beta_w{}^l$. By the construction of ψ_i and ψ_l, $\beta_c{}^i$ and $\beta_w{}^l$ occupy exactly the same places in $\psi_i{}^m$ and $\psi_l{}^p$ and each free variable γ preceding $\beta_c{}^i$ in the $\psi_i{}^m$-order of $\psi_i{}^m$ occurs in $\psi_l{}^p$ in exactly the same places as in $\psi_i{}^m$. By arguments similar to subcase *ii* of Case 1, it can be shown that there is no $\beta_a{}^i$ nor $\beta_b{}^l$ ($a, b = 1, \ldots, j$) which is alphabetically later than $\beta_c{}^i$ and $\beta_w{}^l$ in $\psi_i{}^m$ and $\psi_l{}^p$. So by successive applications of rule I, $\psi_i{}^m$ and $\psi_l{}^p$ can be made repetitious and one of them dropped by rule III. Hence the assumption again fails.

Thus ϕ is always derivable from $\ulcorner \psi_1 V \psi_2 V \ldots V \psi_n \urcorner$ by rules I, II and III.

PART B: *If there is no n such that* $\ulcorner \psi_1 V \psi_2 V \ldots V \psi_n \urcorner$ *is a tautology, then* $\ulcorner -\phi \urcorner$ *has a true interpretation in the domain of positive integers.*

Proof: Since in each ψ_i ($i = 1, \ldots$) every $\beta_c{}^i$ ($c = 1, \ldots, j$), which is preceded in ψ_i-order by the variables $\alpha_f{}^i$ ($f = 1, \ldots, k$), is a function of all these variables, and every $\beta_d{}^i$ ($d = 1, \ldots, j$) which is not preceded by any $\alpha_q{}^i$ ($q = 1, \ldots, k$) is the same as $\beta_d{}^1$, Gödel's argument for the special case when ϕ is in Skolem normal form is at once extensible to the general case when ϕ is in any prenex normal form. q. e. d.

An immediate corollary of Theorem I is Herbrand's result that a prenex schema ϕ is demonstrable if and only if a Herbrand Tautology is constructible for ϕ. Indeed, Part A above is nothing but a simple reformulation, adapted from Hilbert and Bernays,[6] of Herbrand's proof of the sufficiency of a Herbrand Tautology for the demonstrability of ϕ. Part B, however, in showing that such a tautology is a necessary condition for demonstrability, differs essentially from Herbrand's argument in spirit as well as in detail. For, on the one hand, the latter argument is interpretable as presenting a method for actually calculating a number n, whenever a proof of ϕ is given, such that ψ_n is the last alternand of a con-

structible Herbrand Tautology; on the other hand Part B is a non-finitistic argument turning on either set-theoretic notions or the law of excluded middle for whole numbers, as may easily be observed by considering either Gödel's original proof[7] or Hilbert and Bernay's restatement of it.[8]

The distinction between the arguments of Herbrand and Gödel reflects, of course, a distinction as to what are thought to be proper meta-mathematical questions. To Herbrand the notions of validity and satisfiability, being set-theoretic, have no proper meaning. Hence the notion of completeness in the Gödel sense has no meaning, and the question as to whether quantification theory is complete in this sense is an illegitimate question. The corresponding proper finitistic question is: "What are the criteria for quantificational provability?" This Herbrand has answered. Certain passages in Herbrand's Thesis[9] (notably passages on pages 110, 118, 119) might give the impression that Herbrand is treating questions of satisfiability. Closer reading, however, discloses that Herbrand is redefining the set-thoretic notions of validity and satisfiability so as to give them a finitistic meaning. Consequently, his apparent proof of completeness, obtained by combining the theorem on page 111 with the theorem on page 112, omits the crucial step of showing that an infinite satisfying truth-assignment is obtainable (in a non-finitistic sense), not because of an oversight but because it lacks finitistic meaning.

A second corollary of Theorem I and again a result of Herbrand is the following:[10]

Every actually given demonstration of a prenex schema ϕ is mechanically transformable into a Herbrand Normal Proof of ϕ.

Proof: The constructing of an ordered series of alternations of the forms $\psi_1, \ulcorner \psi_1 V \psi_2 \urcorner, \ldots, \ulcorner \psi_1 V \psi_2 V \ldots V \psi_n \urcorner, \ldots$, where each ψ_i $(i = 1, \ldots)$ is obtained from ϕ as in Theorem I, and the testing of each such alternation as it is constructed to see whether it is a tautology are always mechanical processes. Since a proof of ϕ is given, ϕ is valid. Hence by Theorem I, the process of constructing and checking alternations is finite and ends in the discovering of a Herbrand Tautology from which ϕ is mechanically derivable by rules I, II, and III. q. e. d.

Worth noting is that the method of transformation outlined above may be formalized as a general recursive function whereas the method given by Herbrand is a primitive recursive function.

Finally, it is quite obvious that both Theorem I and the two corollaries are extensible to the multi-valued quantification theories studied by Rosser and Turquette.[11]

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[1] Herbrand, J., "Recherches sur la theorie de la demonstration," *Trav. Soc. Sci. et Lettres Varsovie*, Cl. III, 1930, pp. 1–128.

[2] Gödel, K., "Die Vollstandigkeit der Axiome des logischen Funktionenkalkuls," *Mh. Math. Physik*, **37**, 349–360 (1930).

[3] Hilbert and Bernays, *Grundlagen der Mathematik*, Vol. I and II, Berlin, 1934, 1939.

[4] The notation is Quine's in his paper "On Natural Deduction," *J. Symbolic Logic*, **15**, 93–102 (1950).

[5] The complexity of this construction as compared with Gödel's is due, of course, to the difference between any prenex form and a Skolem form.

[6] Hilbert and Bernays, op. cit. See especially the discussion of "disparates Funktionensystem," pp. 167, 168, Vol. II.

[7] Gödel, op. cit.

[8] Hilbert and Bernays, op. cit., pp. 185–189.

[9] Herbrand, op. cit.

[10] Herbrand, op. cit.

[11] Rosser, J. B., and Turquette, A. R., "Axiom Schemes for *m*-Valued Functional Calculi of First Order, Part II," *J. Symbolic Logic*, **16**, 22–34 (1951).

*THE SCHRÖDINGER FINAL AFFINE FIELD LAWS**

By Václav Hlavatý

Graduate Institute for Applied Mathematics, Indiana University, Bloomington, Indiana

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1. The Problem to Solve.—Denote by $\Gamma^{\nu}_{\lambda\mu}$ an arbitrary connection. It gives rise to three principal tensors namely "the tensor of torsion"

$$S_{\lambda\mu}{}^{\nu} = \Gamma^{\nu}_{[\lambda\mu]},$$

the "tensor of curvature"

$$P_{\omega\mu\lambda}{}^{\nu} = \frac{\partial}{\partial x^{\mu}} \Gamma^{\nu}_{\lambda\omega} - \frac{\partial}{\partial x^{\omega}} \Gamma^{\nu}_{\lambda\mu} + \Gamma^{\nu}_{\alpha\mu} \Gamma^{\alpha}_{\lambda\omega} - \Gamma^{\nu}_{\alpha\omega} \Gamma^{\alpha}_{\lambda\mu}$$

and the "contracted tensor of curvature"

$$P_{\mu\lambda} = P_{\alpha\mu\lambda}{}^{\alpha}.$$

These tensors will play a leading role in the following considerations.

In his paper "The Final Affine Field Laws, I" (*Proc. Roy. Irish Acad.*, **51**, 168–171 (1947)), E. Schrödinger bases his unified theory of relativity on a variational principle which leads to 68 equations

$$\frac{\partial}{\partial x^{\omega}} g_{\lambda\mu} = \Gamma^{\alpha}_{\lambda\omega} g_{\alpha\mu} + \Gamma^{\alpha}_{\omega\mu} g_{\lambda\alpha} \tag{1a}$$

$$S_{\lambda\alpha}{}^{\alpha} = 0 \tag{1b}$$

where

$$g_{\lambda\mu} = P_{\lambda\mu} + F_{\lambda\mu} \tag{1c}$$

$$F_{\lambda\mu} = \frac{2}{3}\left(\frac{\partial}{\partial x^\lambda} F_\mu - \frac{\partial}{\partial x^\mu} F_\lambda\right) \tag{2}$$

and F_μ is a vector.

Substituting from (1c) into (1a) one obtains a system (1a, b) of 68 partial differential equations of the second order for 68 unknowns $\Gamma^\nu_{\lambda\mu}$ and F_μ.

The main scope of this paper is to reduce this system of 68 equations to another system of 20 partial differential equations for 20 unknowns $g_{\lambda\mu}$, F_μ and to find explicit solution $\Gamma^\nu_{\lambda\mu}$ in tensorial form of (1) in terms of $g_{\lambda\mu}$ and F_μ. However, we will not confine ourselves to the case $n = 4$. Our results in sections 2 and 3 are valid for any $n > 1$. In section 4 we are giving the "first approximation" of the Schrödinger connection for the case $n = 4$.

2. *Preliminary Results.*—We are introducing here without proofs[1] some results about the system (1a, b) which we will need in the next section. Put

$$g_{(\lambda\mu)} = h_{\lambda\mu},\ g_{[\lambda\mu]} = k_{\lambda\mu}$$

and assume that the determinants g and h of $g_{\lambda\mu}$ and $h_{\lambda\mu}$, respectively, are of rank n. Then the system (1a) is equivalent to[2]

$$\Gamma^\nu_{\lambda\mu} = \left\{{\nu \atop \lambda\mu}\right\} + S_{\lambda\mu}{}^\nu + 2h^{\alpha\nu}S_{\alpha(\mu}{}^\beta k_{\lambda)\beta}. \tag{3}$$

The tensor $S_{\lambda\mu}{}^\nu$ satisfies the condition

$$2S_{\xi\eta\zeta}X^{\xi\eta\zeta}_{\omega\mu\nu} = K_{\omega\mu\nu} \tag{4}$$

where

$$K_{\omega\mu\nu} = \nabla_\omega k_{\nu\mu} + \nabla_\mu k_{\omega\nu} + \nabla_\nu k_{\omega\mu} = K_{[\omega\mu]\nu} \tag{5a}$$

$$X^{\xi\eta\zeta}_{\omega\mu\nu} = \delta^{[\xi}_{[\omega}\delta^{\eta]}_{\mu]}\delta_\nu{}^\zeta - 2\delta^{[\xi}_\nu k_{[\omega}{}^{\eta]}k_{\mu]}{}^\zeta - 2\delta^{[\xi}_{[\mu}k_{\omega]}{}^{\eta]}k_\nu{}^\zeta \tag{5b}$$

and $\delta_\beta{}^\alpha$ is the Kronecker delta. Let us introduce the symbols

$$'X^{ab\zeta}_{cd\nu} = 2X^{ab\zeta}_{cd\nu}, \qquad a < b \quad c < d$$

the remaining $'X^{\xi\eta\zeta}_{\omega\mu\nu} = 0$ and denote by $'X$ the determinant of $\frac{n^2(n-1)}{2}$ rows whose arbitrary row (column) contains all elements $'X^{ab\zeta}_{cd\nu}$ with fixed c, d, ν (with fixed a, b, ξ). Assuming $'X \neq 0$ we introduce the tensor $Y^{\alpha\beta\gamma}_{\omega\mu\nu}$ by means of the conditions

$$2Y^{cd\alpha}_{ab\beta} = -2Y^{dc\alpha}_{ab\beta} = 2Y^{dc\alpha}_{ba\beta} = -2Y^{cd\alpha}_{ba\beta} = \frac{\partial \ln {}'X}{\partial {}'X^{ab\beta}_{cd\alpha}} \tag{6}$$

and the remaining $Y^{\alpha\beta\gamma}_{\omega\mu\nu} = 0$. Then (5a) yields

$$2S_{\omega\mu\nu} = K_{\alpha\beta\gamma}Y^{\alpha\beta\gamma}_{\omega\mu\nu} \tag{7}$$

and this is *the only solution* of (4). Substituting it into (3) we obtain

$$\Gamma^{\nu}_{\lambda\mu} - \left\{{}^{\nu}_{\lambda\mu}\right\} = V_{\lambda\mu}{}^{\nu} \tag{8a}$$

where

$$2V_{\lambda\mu}{}^{\nu} = (2\nabla_{(\alpha}g_{\gamma)\beta} + \nabla_{\beta}g_{\alpha\gamma})\,Y^{\alpha\beta\gamma}_{\xi\eta\zeta}(h^{\zeta\nu}\delta^{\xi}_{\lambda}\delta^{\eta}_{\mu} + 2h^{\xi\nu}\delta^{\eta}_{(\lambda}k_{\mu)}{}^{\zeta}). \tag{8b}$$

These equations express the solution of (1a) *in tensorial form*, in terms of the tensor $g_{\lambda\mu}$ only. Substituting from (8) into $P_{\lambda\mu}$ we obtain

$$P_{\mu\lambda} = H_{\mu\lambda} + \nabla_{\mu}V_{\lambda\alpha}{}^{\alpha} - \nabla_{\alpha}V_{\lambda\mu}{}^{\alpha} + V_{\beta\mu}{}^{\alpha}V_{\lambda\alpha}{}^{\beta} - V_{\beta\alpha}{}^{\alpha}V_{\lambda\mu}{}^{\beta} \tag{9a}$$

where $H_{\mu\lambda} = H_{\lambda\mu} = H_{\alpha\mu\lambda}{}^{\alpha}$ is the contracted curvature tensor belonging to the curvature tensor $H_{\omega\mu\lambda}{}^{\nu}$ of $\left\{{}^{\nu}_{\lambda\mu}\right\}$ The equation (9a) may be simplified by means of

$$V_{\alpha\mu}{}^{\alpha} = V_{\mu\alpha}{}^{\alpha} = \frac{1}{2}\frac{\partial}{\partial x^{\mu}}\ln g/h \tag{9b}$$

(resulting from (1b)).

3. *The Solution of the Problem.*—The equations (1b) and (7) yield

$$h^{\mu\nu}Y^{\alpha\beta\gamma}_{\omega\mu\nu}K_{\alpha\beta\gamma} = 0.$$

Substituting into this equation for $K_{\alpha\beta\gamma}$ from (5a) one obtains by virtue of (1c)

$$h^{\mu\nu}Y^{\alpha\beta\gamma}_{\omega\mu\nu}(\nabla_{\alpha}P_{[\gamma\beta]} + \nabla_{\beta}P_{[\alpha\gamma]} + \nabla_{\gamma}P_{[\alpha\beta]} + \nabla_{\alpha}F_{\gamma\beta} + \nabla_{\beta}F_{\alpha\gamma} + \nabla_{\gamma}F_{\alpha\beta}) = 0. \tag{10a}$$

The first term on the left-hand side is identically equal to zero, while the sum of the three last terms in the brackets is equal to $2\nabla_{\gamma}F_{\alpha\beta}$. Hence (10a) reduces to

$$3h^{\mu\nu}Y^{\alpha\beta\gamma}_{\omega\mu\nu}(\nabla_{\beta}P_{[\alpha\gamma]} + \nabla_{\gamma}P_{[\alpha\beta]}) = 8h^{\mu\nu}Y^{\alpha\beta\gamma}_{\omega\mu\nu}\nabla_{\gamma}\nabla_{[\beta}F_{\alpha]} \tag{10b}$$

Substituting into (10b) from (9) *we obtain n partial differential equations (for n unknowns F_{μ})*, where all the *coefficients are functions of $g_{\lambda\mu}$ and its derivatives only.*

If we denote by F_{μ} the solution of this system then we obtain from (1c), (2), and (9a)

$$g_{\mu\lambda} - \frac{2}{3}\left(\frac{\partial}{\partial x^{\mu}}F_{\lambda} - \frac{\partial}{\partial x^{\lambda}}F_{\mu}\right) = H_{\mu\lambda} + \nabla_{\mu}V_{\lambda\alpha}{}^{\alpha} - V_{\alpha}V_{\lambda\mu}{}^{\alpha} + V_{\beta\mu}{}^{\alpha}V_{\lambda\alpha}{}^{\beta} - V_{\beta\alpha}{}^{\alpha}V_{\lambda\mu}{}^{\beta}. \tag{11}$$

Substituting into (11) from (8b) *we obtain n^2 partial differential equations for n^2 unknowns $g_{\lambda\mu}$. Substituting the solution $g_{\lambda\mu}$ into (8b) we have in (8a) the explicit solution of the system (1a, b) in tensorial form in terms of $g_{\lambda\mu}$ and F_{μ}.*

4. *First Approximation for $n = 4$.*—In this section we are going to

investigate the previous results in the "first approximation," i.e., in neglecting all products of $r > 1$ components $k_{\lambda\mu}$. The equality sign for the first approximation will be denoted by $\doteq$. Moreover, we assume $n = 4$.

First of all we have by virtue of (5), (6) and (9b)

$$X^{\xi\eta\zeta}_{\omega\mu\nu} \doteq Y^{\xi\eta\zeta}_{\omega\mu\nu} \doteq \delta^{[\xi}_{[\omega}\delta^{\eta]}_{\mu]}\delta^{\zeta}_{\nu} \tag{12a}$$

$$V_{\lambda\mu}{}^{\nu} \doteq V_{[\lambda\mu]}{}^{\nu} \doteq \frac{1}{2}K_{\lambda\mu}{}^{\nu} \tag{12b}$$

$$V_{\lambda\alpha}{}^{\alpha} = V_{\alpha\lambda}{}^{\alpha} \doteq \frac{1}{2}K_{\lambda\alpha}{}^{\alpha} = \nabla_{\alpha}k_{\lambda}{}^{\alpha} \doteq 0 \tag{12c}$$

so that we obtain from (11)

$$h_{\mu\lambda} \doteq H_{\mu\lambda}. \tag{13a}$$

This system of 10 partial differential equations of second order for 10 unknowns $h_{\lambda\mu}$ admits a solution known from the time of Riemann, namely *the solution $h_{\lambda\mu}$ with constant curvature $\doteq 4$.*[3] This solution satisfies also the equation

$$-3H_{\omega\mu\lambda}{}^{\nu} \doteq h_{\omega\lambda}\delta_{\mu}{}^{\nu} - h_{\mu\lambda}\delta_{\omega}{}^{\nu}. \tag{13b}$$

Hence we know already the symmetrical part $h_{\lambda\mu}$ of $g_{\lambda\mu}$ so that we know also the Christoffel symbols $\{{}^{\nu}_{\lambda\mu}\}$ and ∇_{μ}.

Next we will consider the system (10b) of four partial differential equations for F_{μ}. First of all we have by virtue of (5a), (9a), (12b, c) and (13b)[4]

$$P_{[\mu\lambda]} \doteq \frac{1}{2}\nabla^{\alpha}K_{\mu\lambda\alpha} \doteq \frac{1}{2}\Delta k_{\mu\lambda} + \frac{2}{3}k_{\lambda\mu}$$

$$\Delta = h^{\lambda\mu}\nabla_{\lambda}\nabla_{\mu}. \tag{14}$$

Consequently (10b) reduces to

$$3\nabla^{\alpha}\Delta k_{\omega\alpha} \doteq 4\left(\Delta F_{\omega} - \frac{\partial}{\partial x^{\omega}}\nabla_{\alpha}F^{\alpha} + F_{\omega}\right). \tag{15}$$

This equation may be simplified. If we interchange the operators ∇^{α} and $\nabla_{\mu}\nabla_{\lambda}$ in

$$\nabla^{\alpha}\Delta k_{\alpha\omega} = h^{\mu\lambda}\nabla^{\alpha}\nabla_{\mu}\nabla_{\lambda}k_{\alpha\omega}$$

by means of the formula used in the footnote 4 we obtain by virtue of (12c)

$$\nabla^{\alpha}\Delta k_{\omega\alpha} \doteq 0. \tag{16a}$$

Moreover, if we use the well-known identity

$$\nabla_{\omega}\nabla_{\mu}F_{\lambda} + \nabla_{\mu}\nabla_{\lambda}F_{\omega} + \nabla_{\lambda}\nabla_{\omega}F_{\mu} = 0$$

(valid for any arbitrary vector) then we obtain by virtue of (13a)

$$\Delta F_\omega \doteq F_\omega - 2\frac{\partial}{\partial x^\omega}\nabla_\alpha F^\alpha. \tag{16b}$$

The equations (16) reduce (15) to

$$F_\omega \doteq \frac{3}{2}\frac{\partial}{\partial x^\omega}\nabla_\alpha F^\alpha. \tag{17a}$$

Hence F_ω is in the first approximation a gradient so that we have according to (2)

$$F_{\lambda\mu} \doteq 0. \tag{17b}$$

Therefore we obtain from (1c) by virtue of (14) (or from (11) by virtue of (12))

$$\frac{5}{3}k_{\lambda\mu} - \frac{1}{2}\Delta k_{\lambda\mu} \doteq 0. \tag{18}$$

This is a system of six partial differential equations for six unknowns $k_{\lambda\mu}$. Using the fact that $n = 4$ we may transform it in a simpler form. First of all we have for $n = 4$

$$\nabla_\omega k_{\mu\lambda} + \nabla_\mu k_{\lambda\omega} + \nabla_\lambda k_{\omega\mu} \equiv \epsilon_{\omega\mu\lambda\nu}\rho u^\nu \tag{19a}$$

where $\epsilon_{\omega\mu\lambda\nu}$ is the four vector density of weight $+1$ with the components $+1, -1, 0$, ρ is a scalar density of weight -1 and u^ν is a contravariant vector. From (19a) we obtain at once[5]

$$\nabla_\alpha \rho u^\alpha \equiv 0. \tag{19b}$$

From (19a) we have by virtue of (13a, b)

$$\Delta k_{\lambda\mu} \doteq \epsilon_{\alpha\beta\lambda\mu}\nabla^\alpha \rho u^\beta + \frac{4}{3}k_{\mu\lambda}. \tag{20}$$

The equations (20) and (18) yield

$$\frac{14}{3}k_{\mu\lambda} \doteq \epsilon_{\mu\lambda\alpha\beta}\nabla^\alpha \rho u^\beta \tag{21}$$

and this equation leads by virtue of (19) to

$$14\ \epsilon_{\omega\mu\lambda\nu}\rho u^\nu \doteq 9\epsilon_{\alpha\beta[\omega\mu}\nabla_{\lambda]}\nabla^\alpha \rho u^\beta. \tag{22a}$$

Multiplying this equation by the contravariant four vector density $\epsilon^{\omega\mu\lambda\gamma}$ of weight -1 whose components are $+1$, -1, 0 one obtains

$$7\ \rho u^\gamma - 3\ \nabla_\alpha \nabla^{[\alpha}\rho u^{\gamma]} \doteq 0 \tag{22b}$$

and this equation reduces by virtue of (19b) and (13a) to

$$3\,\Delta\rho u^{\nu} \doteq 11\,\rho u^{\nu}. \tag{22c}$$

We have obtained in (22c) four partial differential equations of second order for four unknowns ρu^{ν}. Substituting the solution ρu^{ν} of this system in (21) we finally obtain $k_{\lambda\mu}$. Substituting this tensor in (8a) we obtain by virtue of (12b) and (5a)

$$\Gamma^{\nu}_{\lambda\mu} - \left\{{\nu \atop \lambda\mu}\right\} \doteq \frac{1}{2}\left(\nabla_{\lambda}k^{\nu}{}_{\mu} + \nabla_{\mu}k_{\lambda}{}^{\nu} + \nabla^{\nu}k_{\lambda\mu}\right). \tag{23}$$

To explain these results once in detail let us summarize: *Denote by* $h_{\lambda\mu} = h_{(\lambda\mu)}$ *the tensor of rank 4 and curvature 4. Let* $\left\{{\nu \atop \lambda\mu}\right\}$ *be its corresponding Christoffel symbols and* ∇_{μ} *the symbol of the corresponding covariant derivative. Let* ρu^{ν} *be a solution of (22c) and define* $k_{\lambda\mu}$ *by (21). Then the first approximation of the solution* $\Gamma^{\nu}_{\lambda\mu}$ *of the Schrödinger system (1a, b, c) is given by (23), while*

$$F_{\lambda\mu} \doteq 0, \tag{24a}$$

$$\nabla^{\alpha}k_{\alpha\mu} \doteq 0. \tag{24b}$$

Remark I: If we substitute into the *identities* (19) the solution ρu^{ν} we obtain the *equations*

$$\nabla_{\omega}k_{\mu\lambda} + \nabla_{\mu}k_{\lambda\omega} + \nabla_{\lambda}k_{\omega\mu} \doteq \epsilon_{\omega\mu\lambda\nu}\rho u^{\nu} \tag{25a}$$

$$\nabla_{\alpha}\rho u^{\alpha} \doteq 0. \tag{25b}$$

In both of these equations (as well as in (21)) the tensor $h_{\lambda\mu}$ is involved through the solution ρu^{ν} of (22c). If we introduce the bivector density $2f^{\lambda\nu} = \epsilon^{\omega\mu\lambda\nu}k_{\omega\mu}$ then the equations (24b) and (25a) expressed in terms of $f^{\lambda\nu}$ are

$$\nabla_{\omega}f_{\mu\nu} + \nabla_{\mu}f_{\nu\omega} + \nabla_{\nu}f_{\omega\mu} \doteq 0$$

$$\nabla_{\alpha}f^{\alpha\nu} \doteq \rho u^{\nu}.$$

These equations have the familiar form of Maxwell equations while (25b) reminds us of the equation of continuity.

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[1] The proofs will be given in my paper "The Elementary Basic Principles of the Unified Theory of Relativity B" to be published in the *Journal of Rational Mechanics and Analysis*. Some of the results of this section have been used also in my papers "The Elementary Basic Principles of the Unified Theory of Relativity," PROC. NATL. ACAD. SCI., **38,** 243–247 (1952), and "The Einstein Connection of the Unified Theory of Relativity," *Ibid.*, **38,** 415–419 (1952).

[2] The lowering and/or raising of indices is based on $h_{\lambda\mu}$ and its contravariant components $h^{\alpha\nu}$. The symbols $\left\{{\nu \atop \lambda\mu}\right\}$ in reference 3 are Christoffel symbols belonging to $h_{\lambda\mu}$. The symbol ∇_{μ} used later on denotes the covariant derivative with respect to $\left\{{\nu \atop \lambda\mu}\right\}$.

[3] A necessary and sufficient condition for the existence of another solution ${}'h_{\lambda\mu} \neq h_{\lambda\mu}$ of (13a) which has the rank 4 is

$$\nabla_\alpha T_{\mu\lambda}{}^\alpha = \frac{1}{2}\left(\nabla_\mu \frac{\partial}{\partial x^\lambda} \ln \frac{h}{{}'h} + T_{\mu\lambda}{}^\alpha \frac{\partial}{\partial x^\alpha} \ln \frac{h}{{}'h}\right) - T_{\alpha\mu}{}^\beta T_{\beta\lambda}{}^\alpha.$$

Here

$$T_{\mu\lambda}{}^\nu = T_{\lambda\mu}{}^\nu = \left\{{}^{\nu}_{\lambda\mu}\right\} - {}'\left\{{}^{\nu}_{\lambda\mu}\right\}$$

while ${}'h$ and ${}'\left\{{}^{\nu}_{\lambda\mu}\right\}$ are the determinant and the Christoffel symbols of ${}'h_{\lambda\mu}$.

[4] We are using here the known identity

$$2\,\nabla_{[\omega}\nabla_{\mu]} k_{\lambda\nu} = H_{\omega\mu\lambda}{}^\alpha k_{\alpha\nu} + H_{\omega\mu\nu}{}^\alpha k_{\lambda\alpha}$$

as well as (13b).

[5] In both identities (19) the symbol ∇ may be replaced by $\dfrac{\partial}{\partial x}$ so that the connection $\left\{{}^{\nu}_{\lambda\mu}\right\}$ is not involved.

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*DEGREE OF APPROXIMATION ON A JORDAN CURVE**

BY J. L. WALSH AND H. MARGARET ELLIOTT

HARVARD UNIVERSITY AND WASHINGTON UNIVERSITY

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A number of recent papers have treated the question of degree of approximation on a Jordan curve either by polynomials or other rational functions, or by functions analytic and bounded in a given region. It is the object of the present paper to unify these results, especially by use of general moduli of continuity, degree of approximation, and approximating functions. Our methods and results generalize the classical theory developed by de la Vallée Poussin[1] on trigonometric approximation to periodic functions of a real variable; some of the present results (e.g., Theorems 2 and 3) are novel even in this special situation.

THEOREM 1. *Let C be an analytic Jordan curve, which lies in a region D. Let the sequence of functions $f_n(z)$ be analytic in D and satisfy the inequalities* $(R > 1)$

$$|f_n(z)| \leqq A_1 R^n, \qquad z \text{ in } D, \tag{1}$$

$$|f(z) - f_n(z)| \leqq A_2 \Omega(n)/n^k, \qquad z \text{ on } C, \tag{2}$$

for some function $f(z)$. If $\omega(\delta)$ denotes the modulus of continuity of $f^{(k)}(z)$ on C, then

$$\omega(\delta) \leqq A_3\{\delta \int_a^{a/\delta} \Omega(x)\,dx + \int_{1/\delta}^{\infty} [\Omega(x)/x]\,dx\}, \; 0 < \delta \leqq 1/a, \tag{3}$$

where a is a suitably chosen positive constant.

Here and below all numbers A_j represent positive constants which may vary from one theorem and its proof to another but which are independent of n, z, and δ; k denotes a non-negative integer; $f^{(0)}(z)$ indicates $f(z)$ itself; $\Omega(x)$ is assumed to be a real, non-negative function which is non-increasing (i.e., as x increases) for x sufficiently large and which is such that $\int^{\infty}[\Omega(x)/x]\,dx$ exists.

Under an analytic one-to-one transformation, the modulus of continuity of the transformed function (or any of its derivatives) is less than a constant times the modulus of continuity of the original function (or its corresponding derivative);[2] if an arbitrary C is given, a map of the interior of C onto the interior of the unit circle γ maps an annulus containing C onto an annulus containing γ; inequality (2) valid on C implies (2) valid on γ, and (1) valid in D implies (1) valid in some region containing γ; thus it is sufficient (as we do) to prove Theorem 1 for the case that C is the unit circle. Use of the Laurent development shows[3] that there exist polynomials $S_{n,N}(z)$ in z and $1/z$ of respective degrees N such that

$$|f_n(z) - S_{n,N}(z)| \leqq A_4R^n/\rho^N, \qquad z \text{ on } C, \tag{4}$$

where $\rho > 1$. Choose the positive integer λ so that $R/\rho^{\lambda} = r < 1$. Then (2) and (4) yield

$$|f(z) - S_{n,\lambda n}(z)| \leqq A_2\Omega(n)/n^k + A_4r^n, \qquad z \text{ on } C.$$

Set $S_{n,\lambda n+j}(z) \equiv S_{n,\lambda n}(z)$ for $j = 0, 1, \ldots, \lambda - 1$. Since $r^n \leqq A_5/n^{k+2}$ and $\Omega(n) \leqq \Omega((\lambda n + j)/2\lambda)$ for n sufficiently large, $j = 0, 1, \ldots, \lambda - 1$, we have

$$|f(z) - S_{n,\lambda n+j}(z)| \leqq A_6[\Omega((\lambda n + j)/2\lambda) + 1/(\lambda n + j)^2]/(\lambda n + j)^k, \; z \text{ on } C.$$

Hence it follows[1, 4] on separating $f(z)$ into its real and imaginary parts that

$$\omega(\delta) \leqq A_7\{\delta\int_a^{a/\delta}[\Omega(x/2\lambda) + x^{-2}]\,dx + \int_{1/\delta}^{\infty}[\Omega(x/2\lambda)/x + x^{-3}]\,dx\}, \; 0 < \delta \leqq 1/a$$

where $a > 2\lambda$ and $\Omega(x)$ is non-increasing for $x > a/2\lambda$. With suitable change of variable and simplification, we obtain an inequality of form (3).

Theorem 1 includes the case of approximation to $f(z)$ on C by polynomials $f_n(z)$ in z of respective degrees n on C; here (2) implies (1) for z in an arbitrary bounded region containing C, where R depends on D. Theorem 1 similarly includes the case of approximation to a function $f(e^{i\theta})$ of period 2π on $-\infty < \theta < \infty$ by trigonometric polynomials $f_n(e^{i\theta})$ of respective degrees n, for we can interpret C as the unit circle and the $f_n(e^{i\theta})$ as polynomials in z and $1/z$; again (2) implies (1) in an annular region containing C.

We establish next an extension of Privaloff's well-known result[5] on conjugate functions:

THEOREM 2. *Let $f(\theta)$, periodic with period 2π, be continuous with modulus of continuity $\omega(\delta)$; let $g(\theta)$ be conjugate to $f(\theta)$. Then*

$$|g(\theta + h) - g(\theta)| \leqq \omega_0(h) \equiv A_1 \left[\int_0^h \frac{\omega(t)}{t}\, dt + h \int_h^\pi \frac{\omega(t)}{t^2}\, dt \right], \quad h > 0, \tag{5}$$

provided $\int_0^h [\omega(t)/t]\, dt$ exists.

We note that some restriction on $\omega(\delta)$ is necessary here, for the mere continuity of $f(\theta)$ does not imply the continuity of $g(\theta)$.

Let $h > 0$. We have[6]

$$g(\theta + h) - g(\theta) = -\frac{1}{2\pi}\left(\int_{-\pi}^{-2h} + \int_{2h}^{\pi} \right) [f(\theta + t) - f(\theta)] \times \left[\operatorname{ctn} \frac{1}{2}(t - h) - \operatorname{ctn} \frac{1}{2}t \right] dt + \frac{1}{2\pi} [f(\theta + h) - f(\theta)] \times \int_2^\pi \left[\operatorname{ctn} \frac{1}{2}(t - h) - \operatorname{ctn} \frac{1}{2}(t + h) \right] dt - \frac{1}{2\pi} \times \int_{-2h}^{2h} \left\{ [f(\theta + t) - f(\theta + h)] \operatorname{ctn} \frac{1}{2}(t - h) - [f(\theta + t) - f(\theta)] \operatorname{ctn} \frac{1}{2}t \right\} dt$$

Hence since $\operatorname{ctn} \frac{1}{2}(t - h) - \operatorname{ctn} \frac{1}{2}t = \sin \frac{1}{2}h / \sin \frac{1}{2}(t - h) \sin \frac{1}{2}t$, it follows that

$$|g(\theta + h) - g(\theta)| \leqq A_2 \int_{2h}^{\pi} \omega(t) \frac{\sin \frac{1}{2}h}{\sin \frac{1}{2}(t - h) \sin \frac{1}{2}t}\, dt + \omega(h) \times \left[\log \frac{\sin \frac{1}{2}(t - h)}{\sin \frac{1}{2}(t + h)} \right]_{2h}^{\pi} + A_3 \int_{-2h}^{2h} \left[\frac{\omega(|t - h|)}{|t - h|} + \frac{\omega(|t|)}{|t|} \right] dt \leqq A_4 \times \left[\int_0^h \frac{\omega(t)}{t}\, dt + h \int_h^\pi \frac{\omega(t)}{t^2}\, dt \right];$$

$\omega(h)$ is less than twice the last term in the square bracket for $h \leqq 1$.

We remark that the methods of trigonometric approximation developed by de la Vallée Poussin can be used under the present hypothesis; but the conclusion thus obtained is weaker than that of Theorem 2. Nevertheless, those methods are of significance with a hypothesis of degree of convergence; application of a theorem due to Fekete[7] on the modulus of the conjugate of a trigonometric polynomial yields: *if the function $f(\theta)$ of period 2π can be approximated by a trigonometric sum of order n $(n > 0)$ with an error not greater than $\Omega(n)/n^k$, then the conjugate $g(\theta)$ has a continuous derivative of order k with modulus of continuity not greater than*

$$\omega_1(\delta) \equiv A_1 \left[\delta \int_a^{a/\delta} \Omega(x) \log x\, dx + \int_{1/\delta}^{\infty} [\Omega(x) \log x / x]\, dx \right],$$

assuming $\omega_1(\delta)$ to exist and to approach zero with δ.

With a further restriction on $\omega(\delta)$ we prove

THEOREM 3. *Let $f(\theta)$ and $g(\theta)$ satisfy the hypothesis of Theorem 2. If there exists a constant $\gamma > 1$ such that $\omega(\delta)\,|\log\delta|^{\gamma}$ is non-decreasing for δ sufficiently small, then $g(\theta)$ is continuous with modulus of continuity $\omega^*(\delta)$ which satisfies the condition $\omega^*(\delta) \leqq A_2\,|\log\delta|\,\omega(\delta)$.*

There exists a δ_0, $0 < \delta_0 \leqq \frac{1}{2}$, such that $\omega(\delta)\,|\log\delta|^{\gamma}$ is non-decreasing for $0 < \delta \leqq \delta_0$. Without loss of generality we henceforth assume $0 < \delta \leqq \delta_0$.

It follows from Theorem 2 that

$$\omega^*(\delta) \leqq A_3\,[\textstyle\int_0^{\delta} [\omega(t)/t]\,dt + \delta \int_{\delta}^{\pi} [\omega(t)/t^2]\,dt]. \tag{6}$$

We have

$$\int_0^{\delta} \frac{\omega(t)\,|\log t|^{\gamma}}{t\,|\log t|^{\gamma}}\,dt \leqq \omega(\delta)\,|\log\delta|^{\gamma} \int_0^{\delta} \frac{dt}{t\,|\log t|^{\gamma}} \equiv A_4\omega(\delta)\,|\log\delta|; \tag{7}$$

$$\delta\int_{\delta}^{\pi} \frac{\omega(\delta t/\delta)}{t^2}\,dt \leqq \omega(\delta) \int_{\delta}^{\pi} \frac{t+\delta}{t^2}\,dt \leqq A_5\omega(\delta)\,|\log\delta|. \tag{8}$$

We use here the well-known fact that $\omega(\mu\delta) \leqq (\mu+1)\omega(\delta)$, $\mu > 0$. The theorem now follows from inequalities (6)–(8).

We remark that the hypothesis of Theorem 3 on $\omega(\delta)$ is satisfied in particular for $\omega(\delta)$ of the order of the functions δ^{α}, $\delta^{\alpha}\cdot|\log\delta|^{\beta}$, $\delta^{\alpha}\cdot|\log|\log\delta||^{\beta}$, . . ., where $0 < \alpha \leqq 1$ and $\beta > 0$, and of the function $1/|\log\delta|^{\beta}$ where $\beta > 1$.

If $\omega(\delta) = O(1/|\log\delta|\cdot|\log|\log\delta||^{\beta})$, $\beta > 1$, it then follows readily from Theorem 2 that $\omega^*(\delta) \leqq A_3|\log\delta|\cdot|\log|\log\delta|\,|\cdot\omega(\delta)$. Similar conclusions can be drawn for the functions $1/|\log\delta|\cdot|\log|\log\delta|\cdot|\log||\log|\log\delta|||^{\beta}$, $\beta > 1$, ...

We shall use the following known result:[8]

THEOREM 4. *Let the function $U(\theta)$ be of class L^2 on C: $|z| = 1$, let $V(\theta)$ be the function conjugate to $U(\theta)$ on C with $\int_C V(\theta)\,d\theta = 0$, and let $u(z)$ and $v(z)$, respectively, be the functions harmonic interior to C defined by Poisson's integral, taking on the boundary values $U(\theta)$ and $V(\theta)$ almost everywhere on C. Then we have*

$$f_1(z) = \frac{1}{2\pi i}\int_C \frac{U(\theta)\,dt}{t-z} \equiv \frac{1}{2}\,[u(z) + iv(z) + u(0)],\ z \text{ interior to } C,$$

$$f_2(z) = \frac{1}{2\pi i}\int_C \frac{U(\theta)\,dt}{t-z} \equiv -\frac{1}{2}\,[u(1/\bar{z}) - iv(1/\bar{z}) - u(0)],\ z \text{ exterior to } C.$$

The above integrals and all subsequent contour integrals are to be taken counterclockwise. The functions $U(\theta)$ and $V(\theta)$ are not necessarily real.

For the functions $f_1(z)$ and $f_2(z)$ defined in Theorem 4 we prove now

THEOREM 5. *If $U^{(k)}(\theta)$ is continuous on $|z| = 1$ with modulus of continuity $\omega(\delta)$, then $f_1^{(k)}(z)$ and $f_2^{(k)}(z)$ are continuous in $|z| \leqq 1$ and $|z| \geqq 1$, respectively, with moduli of continuity less than $\omega_0(\delta)$, provided $\omega_0(\delta)$ exists and approaches zero with δ.*

The function $V^{(k)}(\theta)$ is conjugate to $U^{(k)}(\theta)$; hence Theorem 2 implies that $V^{(k)}(\theta)$ is continuous with modulus of continuity less than $\omega_0(\delta)$. Since $U(\theta)$ and $V(\theta)$ are continuous on $|z| = 1$, we have $f_1(e^{i\theta}) = (1/2)[U(\theta) + iV(\theta) + u(0)]$, $f_2(e^{i\theta}) = -1/2[U(\theta) - iV(\theta) - u(0)]$. Theorem 5 now follows.[4]

THEOREM 6. *Let C be an analytic Jordan curve. Let $f^{(k)}(z)$ be continuous with modulus of continuity $\omega(\delta)$ on C. Set $f_1(z) = (1/2\pi i) \int_C [f(t)/(t - z)]\, dt$, z interior to C; $f_2(z) = (1/2\pi i) \int_C [f(t)/(t - z)]\, dt$, z exterior to C. Then the boundary values on C of $f_1(z)$ and $f_2(z)$ satisfy the equation $f_1(z) - f_2(z) = f(z)$. On C, $f_1^{(k)}(z)$ and $f_2^{(k)}(z)$ have moduli of continuity less than $\omega_0(\delta)$, provided $\omega_0(\delta)$ exists and approaches zero with δ. If C contains the origin in its interior, then there exist polynomials $p_n(z)$, $P_n(1/z)$ of respective degrees n $(n > 0)$ such that*

$$|f_1(z) - p_n(z)| \leqq A_2\omega_0(1/n)/n^k, \qquad z \text{ in closed interior of } C;$$

$$|f_2(z) - P_n(1/z)| \leqq A_3\omega_0(1/n)/n^k, \qquad z \text{ in closed exterior of } C.$$

Hence there exist polynomials $\pi_n(z, 1/z)$ in z and $1/z$ such that $(n > 0)$

$$|f(z) - \pi_n(z, 1/z)| \leqq A_4\omega_0(1/n)/n^k, \qquad z \text{ on } C.$$

Let $w = \phi(z)$ map the interior of C onto $|w| < 1$, and let $z = \Phi(w)$ denote the inverse function to $\phi(z)$. Then[2] $\partial^k f[\Phi(e^{i\theta})]/\partial\theta^k$ is continuous on $|w| = 1$ with modulus of continuity less than $A_5\omega(\delta)$. By Theorem 4 we have $f[\Phi(w)] = \phi_1(w) - \phi_2(w)$ for $|w| = 1$, where

$$\phi_1(w) = (1/2)[U(w) + iV(w) + U(0)], \qquad |w| \leqq 1,$$

$$\phi_2(w) = -(1/2)[U(1/\bar{w}) - iV(1/\bar{w}) - U(0)], \qquad |w| \geqq 1;$$

here $U(w) + iV(w)$ is the function analytic in $|w| < 1$, continuous in $|w| \leqq 1$, with $U(e^{i\theta}) = f[\Phi(e^{i\theta})]$, $V(0) = 0$.

It follows from Theorem 5 that $\phi_1^{(k)}(w)$ and $\phi_2^{(k)}(w)$ have moduli of continuity on $|w| = 1$ less than $A_6\omega_0(\delta)$.

There exists a closed annulus A in the z-plane bounded by C and by an analytic Jordan curve C' containing C in its interior, which is transformed one-to-one and conformally under the map $w = \phi(z)$. If z is a point of A, we have $\phi_2[\phi(z)] \equiv -\phi_3(z) + \phi_4(z)$, where $\phi_3(z)$ and $\phi_4(z)$ are continuous in A, and

$$\phi_3(z) \equiv -\int_{C'} \frac{\phi_2[\phi(t)]}{t - z}\, dt, \qquad z \text{ interior to } C',$$

$$\phi_4(z) \equiv -\int_C \frac{\phi_2[\phi(t)]}{t-z}\,dt, \qquad z \text{ exterior to } C.$$

On C the function $\phi_3(z)$ is analytic, hence $\phi_3^{(k)}(z)$ satisfies a Lipschitz condition of order unity there. It is easily seen (cf. the proofs given in reference 2) that $d^k\phi_2[\phi(z)]/dz^k$ has modulus of continuity less than $A_7\omega_0(\delta)$ on C. Hence $\phi_4^{(k)}(z)$ has modulus of continuity less than $A_8\omega_0(\delta)$ on C.

We have now $f_1(z) \equiv \phi_1[\phi(z)] + \phi_3(z)$ for z in the closed interior of C, $f_2(z) \equiv \phi_4(z)$ for z in the closed exterior of C; the first part of Theorem 6 follows. The conclusions on the existence of $p_n(z)$ and $P_n(1/z)$ are then a consequence of known results.[9]

THEOREM 7. *In addition to the hypothesis of Theorem 6 suppose there exists a constant $\beta > 1$ such that $\omega(\delta)\,|\log\delta|^\beta$ is non-decreasing for δ sufficiently small. Then in the conclusion of Theorem 6, $\omega_0(\delta)$ may be replaced by $A_1\omega(\delta)\,|\log\delta|$.*

Theorem 7 follows immediately from Theorem 6 and the proof of Theorem 3.

THEOREM 8. *Let C be an analytic Jordan curve containing the origin in its interior; let $f(z)$ be defined on C. Let there exist polynomials in z and $1/z$ of the form $p_n(z) = \sum_{j=-n}^{n} a_{nj}z^j$ such that $(n > 1)$*

$$|f(z) - p_n(z)| \leqq A_1\Omega(n)/n^k, \qquad z \text{ on } C. \tag{9}$$

Set $p_{1n}(z) = \sum_{j=0}^{n} a_{nj}z^j$; $p_{2n}(z) = \sum_{j=-n}^{-1} a_{nj}z^j$; $f_1(z) = (1/2\pi i)\int_C [f(t)/(t-z)]\,dt$, z interior to C; $f_2(z) = -(1/2\pi i)\int_C [f(t)/(t-z)]\,dt$, z exterior to C. Then for z on C

$$|f_\nu(z) - p_{\nu n}(z)| \leqq A_2\left[\frac{\Omega(n)\log n}{n^k} + \int_n^\infty \frac{\Omega(x)\log x}{x^{k+1}}\,dx\right], \qquad \nu = 1, 2, \tag{10}$$

provided the integral on the right converges.

Let N be such that $\Omega(x)$ is non-increasing for $x > N$. For all j such that $2^j > N$ we obtain from inequality (9)

$$|p_{2^{j+1}}(z) - p_{2^j}(z)| \leqq 2A_1\Omega(2^j)/2^{jk}, \qquad z \text{ on } C.$$

Hence[10]

$$|p_{\nu,\,2^{j+1}}(z) - p_{\nu,\,2^j}(z)| \leqq A_3(\log 2^{j+1})\Omega(2^j)/2^{jk}, \qquad \nu = 1, 2, \qquad z \text{ on } C. \tag{11}$$

Let $n > N$ be given. Choose n_0 such that $2^{n_0-1} < n \leqq 2^{n_0}$.

For z on C we have formally (justification follows from the sequel)

$$f_\nu(z) = p_{\nu,2}(z) + \sum_{j=1}^{\infty} [p_{\nu,\,2^{j+1}}(z) - p_{\nu,\,2^j}(z)], \qquad \nu = 1, 2.$$

By the method used to obtain inequality (11), we get

$$|p_{\nu,\,2n_0}(z) - p_{\nu,\,n}(z)| \leqq A_4 (\log 2^{n_0})\Omega(n)/n^k, \qquad \nu = 1, 2, \qquad z \text{ on } C. \tag{12}$$

Inequalities (11) and (12) yield for z on C, supposing the existence of the integrals involved (convergence to $f_\nu(z)$ is a consequence of (9) and Cauchy's integral formula)

$$|f_\nu(z) - p_{\nu,\,n}(z)| \leqq |p_{\nu,\,2n_0}(z) - p_{\nu,\,n}(z)| + \sum_{j=n_0}^{\infty} |p_{\nu,\,2^{j+1}}(z) - p_{\nu,\,2^j}(z)| \leqq$$

$$A_4(\log 2^{n_0})\Omega(n)/n^k + \sum_{j=n_0}^{\infty} (\log 2^{j+1})\Omega(2^j)(2^{j+1} - 2^j)/2^{j(k+1)} \leqq$$

$$A_5[(\log n)\Omega(n)/n^k + \int^{\infty} [(\log x)\Omega(x)/x^{k+1}]\, dx], \qquad \nu = 1, 2.$$

Inequality (10) now follows for all positive n if the constant A_5 is suitably adjusted.

We note that if in particular $\Omega(x) = O(1/x)$, then Theorem 8 enables us to conclude that for z on C $(n > 1)$

$$|f_\nu(z) - p_{\nu,\,n}(z)| \leqq A_6\Omega(n) \log n/n^k, \qquad \nu = 1, 2.$$

If $\Omega(x) = O(1/(\log x)^\beta)$, $\beta > 2$, then for z on C $(n > 1)$

$$|f_\nu(z) - p_{\nu,\,n}(z)| \leqq A_7\Omega(n)\, (\log n)^2/n^k, \qquad \nu = 1, 2.$$

Under a suitable restriction on $\Omega(x)$ we prove

THEOREM 9. *Under the hypothesis of Theorem 8 if there exists a $\beta > 2$ such that $(\log x)^\beta\Omega(x)$ is decreasing for x sufficiently large, then for z on C $(n > 1)$*

$$|f_\nu(z) - p_{\nu,\,n}(z)| \leqq A_8\, (\log n)^2\Omega(n)/n^k, \qquad \nu = 1, 2.$$

With this restriction on $\Omega(x)$ we have

$$\int_n^{\infty} [(\log x)\Omega(x)/x^{k+1}]\, dx \leqq [(\log n)^\beta\Omega(n)/n^k] \int_n^{\infty} (1/x\, (\log x)^{\beta-1})\, dx \leqq$$

$$\frac{(\log n)^\beta\Omega(n)}{n^k} \cdot \frac{1}{(\beta - 2)\, (\log n)^{\beta-2}} \leqq (\log n)^2\Omega(n)/n^k,$$

and Theorem 9 follows at once.

THEOREM 10. *Under the hypothesis of Theorem 8 it follows that $f_1^{(k)}(z)$ and $f_2^{(k)}(z)$ have moduli of continuity less than $\omega_1(\delta)$ as previously defined, assuming $\omega_1(\delta)$ to exist and to approach zero with δ.*

This conclusion follows from (11) by the method of de la Vallée Poussin.[1]

THEOREM 11. *Let C be an analytic Jordan curve, and let the annular region D contain C. Let $f(z)$ be defined on C, and let functions $f_n(z)$ analytic in D satisfy (1) and (2). Then for the components $f_1(z)$, $f_2(z)$ and $f_{1n}(z)$, $f_{2n}(z)$ of $f(z)$ and $f_n(z)$, respectively, we have on C $(n > 1)$*

$$|f_\nu(z) - f_{\nu n}(z)| \leqq A_1\left[(\log n)\frac{\Omega(n)}{n^k} + \int_n^\infty \frac{(\log x)\Omega(x)\,dx}{x^{k+1}}\right], \qquad \nu = 1, 2, \tag{13}$$

provided the integral converges. The moduli of continuity of $f_1^{(k)}(z)$ *and* $f_2^{(k)}(z)$ *satisfy the conclusion of Theorem 10.*

As in reference 3 we have (where we suppose without loss of generality that C contains the origin in its interior)

$$|f_{\nu n}(z) - p_{\nu,\,\lambda n}(z)| \leqq A_2(R/\rho^\lambda)^n, \qquad z \text{ on } C,\ \nu = 1, 2, \tag{14}$$

where $R/\rho^\lambda < 1$, $p_{1,\,\lambda n}(z)$ is a polynomial in z of degree λn, and $p_{2,\,\lambda n}(z)$ is a polynomial in $1/z$ of degree λn.

From (2) and (14) we have

$$|f(z) - [p_{1,\,\lambda n}(z) + p_{2,\,\lambda n}(z)]| \leqq A_3\Omega(n)/n^k, \qquad z \text{ on } C; \tag{15}$$

Theorem 8 implies (10), and (13) follows from (10) and (14). The remainder of Theorem 11 follows from (15) and Theorem 10.

Approximation to a function $f(z)$ in the closure C of a multiply-connected region with hypothesis (1) and (2), where the $f_n(z)$ are analytic in a region D containing C in its interior, implies at once by Cauchy's integral formula the same degree of approximation as (2) in C of any sequence of the respective components of the $f_n(z)$ to the corresponding component of $f(z)$; an inequality of form (1) for the components of the $f_n(z)$ persists in a closed region D_1 interior to D containing C; hence the theorems already established yield results concerning the continuity properties on the boundary of C of the components of $f(z)$. Conversely, if $f(z)$ is analytic interior to C and continuous on C, a given modulus of continuity of $f(z)$ on the boundary of C implies the possibility of suitable approximation on C to the various components of $f(z)$ by properly chosen rational functions, hence implies the corresponding possibility of approximation on C to $f(z)$ by suitable functions $f_n(z)$ satisfying (1) and (2).

If C is an analytic Jordan arc, the results of the present paper combined with methods developed elsewhere[11] yield an analog of Theorem 1 and a result in the converse direction; details are left to the reader.

* The research here presented was accomplished under a contract between Harvard University and the Office of Naval Research. Responsibility for the views expressed lies wholly with the authors.

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*STUDIES ON THE METABOLISM OF PLANT NEOPLASMS, V. AUXIN AS A PROMOTING AGENT IN THE TRANSFORMATION OF NORMAL TO CROWN-GALL TUMOR CELLS**

BY RICHARD M. KLEIN AND GEORGE K. K. LINK

DEPARTMENT OF BOTANY, UNIVERSITY OF CHICAGO

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The cellular alterations involved in the formation of crown-gall tumors in plants have recently aroused renewed interest in associated processes. Because of this, it is important to distinguish between those processes involved in the change of a normal into a tumor cell and those concerned with the subsequent behavior of such a cell. A conceptual and semantic scheme is proposed to serve as a working hypothesis for planning experiments and for relating experimental findings to the general problem of crown-gall genesis.

The techniques and conclusions of Braun and coworkers,[1–3] as well as work under way in the Chicago laboratory, permit an experimental delimitation of several of the processes involved in tumorization (in the limited sense of change of normal cells into primary tumor cells). Braun and Laskaris[1] discovered that an attenuated strain of *Agrobacterium tumefaciens*, the incitant of crown-gall, was incapable of inducing typical tumors in susceptible plants unless the affected tissues received an external supply of either natural (native) or synthetic auxin. They concluded that it was likely that at least two substances were involved in alteration and stimulation of tumor cells, only one of these possibly being auxin. Even earlier, Link, *et al.*,[4] concluded that auxin could be only one entity in a complex of causal factors, all of which were related to tumor formation, and Link and Eggers[5] conclusively demonstrated hyperauxiny in crown-gall tumors and in the contiguous parts of tumor-bearing tomato plants. The subsequent discovery and partial characterization of the action and nature of a "tumor-inducing principle" by Braun[6, 7] demonstrated that at least one of the non-auxinic substances appeared to be a complex molecule of limited heat stability. It is very important for the evaluation of the results of the present study to note that this substance virtually completes its specific action in approximately 36 to 48 hours after introduction of the bacterial

thalli via a wound into the host tissues. On the basis of these reports, it now appears that attenuated strains of crown-gall bacteria are still capable of producing or stimulating the production of Braun's tumor-inducing principle but either lack the ability to produce sufficient, or the right kind of, auxin in the milieu of the plant or to stimulate auxin production by the plant tissues themselves.

With this information at hand, experimental procedures were developed to determine the temporal interrelations between Braun's tumor-inducing principle and auxin, each of which appears to be a necessary, but not a sufficient, causal factor in the genesis of crown-gall tumors. One of the purposes of this paper is to demonstrate that auxin is a promoting substance or cocarcinogen,[8] which acts on the cell while it is in the process of being transformed into a primary tumor cell. (Without auxin a cell which has been altered into an incipient tumor cell through the action of tumor-inducing (incepting) substances, will not be completely transformed into a typical primary tumor cell.) Experiments have also been performed to determine how long incipient tumor cells may remain positively disposed to the promoting action of auxin.

Methods.—A single synthetic auxin, γ-indole-*n*-butyric acid, was used, the choice of this promoting agent being made because of its rapid action.[1] Tomato plants (var. Bonny Best) were grown in 4-inch pots of garden loam in a greenhouse room. When the plants were 30 cm. tall the topmost internode was severed below its node leaving the second apical internode as a stump. All test plants were inoculated with a culture of a single colony isolate of an attenuated strain of crown-gall bacteria (A66) by a needle puncture about 1 cm. below the cut surface. Control plants were similarly bisected but received a sterile puncture in the same site. Only one inoculation or puncture was made in a plant. The inoculated plants and their controls were each divided into two lots. Inoculated and control plants forming the first lot were immediately treated on the cut surface with 0.01 ml. of a 2.5% paste of indole butyric acid in lanolin prepared by the method of Michel.[9] At various time intervals from 0 time to 240 hours after inoculation the auxin paste was removed from the plant by slicing off a 0.2 mm. disk of tissue from the top of the stump. The control and test plants of the second lot received auxin paste at time intervals after inoculation ranging from 0 time to 480 hours. Ten test and ten control plants were used for each time period. As soon as they became visible, axillary buds were removed to minimize the native auxin supply in the tissues. The experiments were terminated 30 days after inoculation.

Experimental Results.—It is evident from figure 1 that the removal of auxin prior to at least 36 to 48 hours after inoculation prevents the appearance of tumors. From 36 hours to 60 hours after inoculation and application of auxin, the eventual size of the tumors appears to be a func-

tion of the length of the auxin presentation time. Under experimental conditions it is difficult to determine exactly when the auxin began its specific promoting action. It is certainly inoperable prior to *at least* 36 hours. However, evidence for the presence of auxin within the tissues after the removal of the auxin-containing paste can be seen in figure 1 where limited auxin tumors formed on treated stumps which had their auxin source removed 24 hours after application. It appears, therefore, that auxin already moved into the tissues may induce some tumor-promoting action.

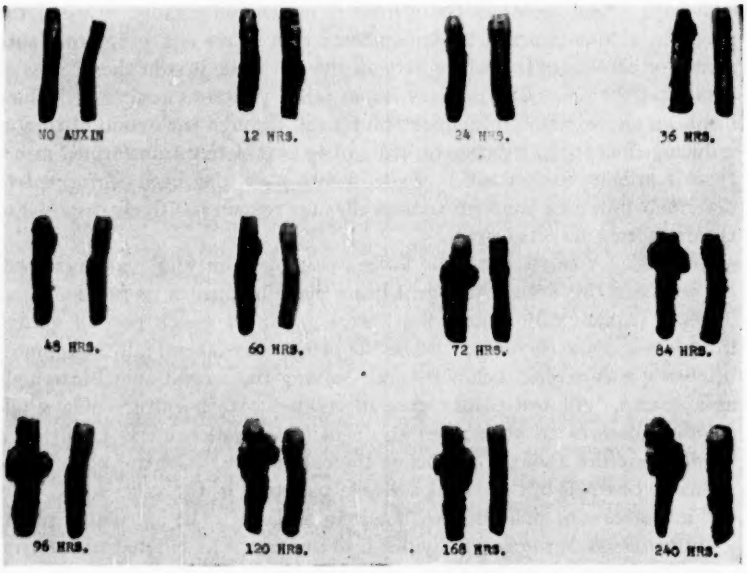

FIGURE 1

Sections of stems from decapitated tomato plants inoculated with the attenuated A66 culture of crown-gall bacterial and treated with auxin at time of inoculation. Auxin-containing paste removed at times indicated under the sections. Inoculated sections at left, control sections at right.

Subsequent to the "60-hour removal" of auxin, there is no further need for supplementary growth substance for there were no significant size differences among tumors formed under conditions of extended auxin presentation time. These findings indicate that a relatively high auxin titer and/or a definite presentation time are required for complete promotion of those cells rendered positively predisposed to promotion by the action of tumor-inducing substance and other, yet unknown, factors.[10]

The incipient tumor cells, once formed, remain positively predisposed

to auxin promotion for some time (figure 2). When auxin paste was applied as late as 10 days after inoculation with attenuated bacteria, the affected cells were transformed into rapidly growing tumor cells. However, when 15 or more days elapsed between inoculation and promotion, the resulting tumors contained root primordia. This probably indicates that the host plant had, to some extent, gained control over the development of the neoplastic tissues. It is of interest that root development from tomato crown-gall tissues has not been observed previously and that this root development did not occur in auxin-treated control material in this study.

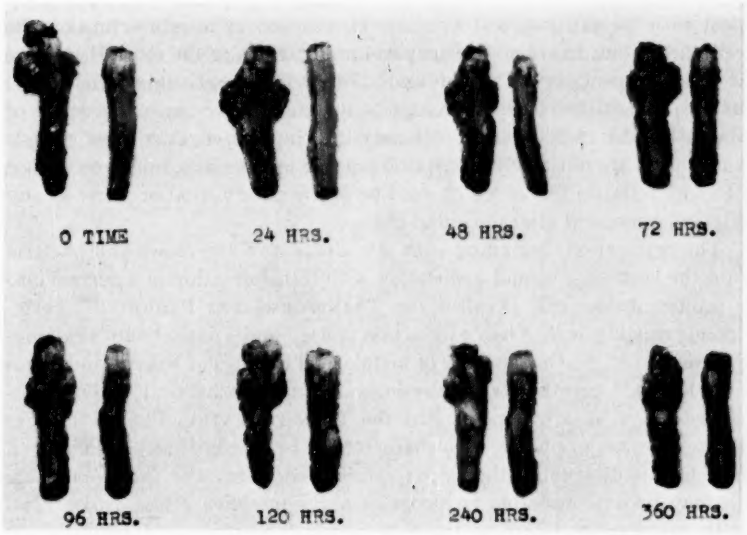

FIGURE 2

Sections of stems from decapitated tomato plants inoculated with attenuated A66 culture of crown-gall bacteria. Auxin-containing paste applied to stumps at times indicated on figure. Inoculated sections at left, control sections at right.

Discussion and Conclusions.—During the last fifty years of research on crown-gall, several attempts have been made to delimit morphological, physiological (including biochemical) and etiological stages in the formation of tumors.[1, 11, 12] Klein,[11] for example, identified three histologically recognizable phases: (1) initiation, ending with the onset of tumor cell multiplication; (2) growth, the phase of rapid tumorous cell proliferation; and (3) maturation, the phase of differentiation of tumor cells into sclerids, etc. Several biochemical changes and activities were associated with these

phases. Consideration of this and previous schemes, together with more recent findings, has necessitated revision and extension of current concepts and semantic schema to deal with the genesis of crown-gall. While this paper deals mainly with the first period of crown-gall formation, the subsequent periods are discussed to show their relation to one another and to the first period.

Genesis of the new entity—a crown gall tumor—from a normal cell or cells can be divided into three periods of cellular change. These periods, when necessary, are subdivided into phases which in turn may be divided into subphases. The terms used here to designate the various periods and phases of tumor cell genesis have (1) etiological implications in that they deal with the intrinsic and extrinsic factors and/or agents acting on the cell under consideration, (2) temporal implications in the sense that there is a time sequence in the periods and (3) material, functional and relational implications in the sense of change or modification of these characters of the cell under consideration. It must be emphasized that these periods and phases are not always sharply separable in time in a tumor composed of many cells, no two of which need be in the same period or phase at any particular moment after the initial change.

The first period, beginning with introduction of the crown-gall bacteria into the host via a wound and ending with transformation of a normal into a primary tumor cell, is called the TRANSFORMATION PERIOD. It corresponds roughly with Klein's induction phase[11] and Tannenbaum's carcinogenesis period.[12] On the basis of histological findings of Riker[13] and Klein and Rasch,[14] together with physiological and biochemical findings discussed below, it now appears that the TRANSFORMATION PERIOD includes at least two major phases, each characterized by a definite series of changes. The first is designated the *Pre-inception (Induction) and Inception Phase* (it may become desirable to recognize a *Pre-inception Phase* as coordinal with an *Inception Phase*) at the close of which no structural modifications are detectable in the prospective tumor cell but during which extensive biochemical changes do occur. These include changes in respiratory levels and pattern, changes in the levels of metabolic intermediates and modifications in the activities of cytoplasmic and nuclear components.[11] These facts leave little doubt that profound changes occur in the affected cells prior to observable structural alterations. Temporally and etiologically, this is the phase during which Braun's tumor-inducing principle is active[6] and during which the cell acquires the potentiality for autonomous behavior but not the capacity for rapid duplication.[10] A cell so incepted, yet histologically and possibly cytologically[15] indistinguishable from non-affected cells, is designated as an *incipient tumor cell*. The incipient tumor cell has become positively predisposed to the action of those other etiological factors and agents which are essential to complete its change into a primary tumor cell.

The second major part of the TRANSFORMATION PERIOD is designated the *Promotion and Completion Phase* (there are indications on etiological grounds that each of these may also have to be recognized as a distinct phase). The etiological factors, processes, and changes occurring during this phase involve a series of cytological, biochemical and physiological changes which are, as yet, incompletely studied. These include cell volume increase, heightened activity of cytoplasmic and nuclear material as judged by changes in staining reactions and biochemical analyses, and a number of altered metabolic activities.[11, 14] It is during this phase that auxin is required as a promoting agent or "cocarcinogen," one of the factors needed to promote, and possibly complete, the change from an incipient to a primary tumor cell. The experiments reported here indicate that these substances are inactive prior to the end of the *Inception Phase*, since the pre-incipient cell is not positively predisposed to the specific promoting action of auxin until it is incepted. There is little doubt that auxin is not the only entity active during the *Promotion and Completion Phase*.

The experimental prolongation of promotion and completion by withholding auxin not only indicates that this substance is specifically required to complete tumorization but also demonstrates that an incipient tumor cell is capable of retaining its positive predisposition to promotion for a considerable length of time. This suggests, but does not prove, that fundamental and presumably permanent alterations were engendered in the incipient tumor cell during the *Pre-induction and Inception Phase*. The application of anti-auxin subsequent to auxin-induced promotion has been shown by de Ropp[17] to be ineffective in preventing tumorization. Klein and Klein,[18] however, did not observe inhibition of tumor formation in tomato plants sprayed with a low concentration of maleic hydrazide, an antiauxin, prior to or concurrently with inoculation by a virulent strain of *A. tumefaciens*. Since there was no way of determining the relative concentrations of auxin and antiauxin at the site of cellular transformation and calculations showed that the antiauxin was in very low concentration within the plant, these latter findings do not invalidate the thesis presented in this report. Attempts to introduce sufficiently large concentrations of maleic hydrazide into tissues being acted upon by attenuated bacteria and auxin were unsuccessful. Further study on this problem is now under way.

Once transformation to an autonomous primary tumor cell has been completed, the cell is ready for the second period of change and behavior, the DUPLICATION PERIOD. During this period a primary tumor cell, as well as its progeny, begins to divide in a rapid but uncoordinated manner. Secondary transformation processes (appositional growth[16]) may also occur during this period when normal cells abutting tumor cells are transformed into tumor cells by agents contributed by primary tumor cells and/or

their progenies. Since Link and Eggers demonstrated hyperauxiny in tissues contiguous to tumors, it may be that these elevated auxin levels play a causal role in appositional tumor genesis. These secondary transformation processes may be different than those involved in primary transformation because the bacteria, and presumably tumor-inducing principle, are no longer required after the primary transformation.[2] The DUPLICATION PERIOD is the time of main increase in the size of the tumorous mass.

With the onset of differentiation of tumor cells into mature elements, tumor sclerids and cells simulating vascular tissue, and with organization of the entire mass of tumor tissue and occluded non-tumor cells into a definite structural, functional, and relational entity, the DIFFERENTIATION AND ORGANIZATION PERIOD begins. Even during this period primary and secondary transformation processes may be in progress and the resultant cells add to the mass of the neoplasm.

* Supported in part by a grant from the Dr. Wallace C. and Clara A. Abbott Memorial Fund of the University of Chicago and a fellowship to the senior author from the American Cancer Society recommended by the Committee on Growth, National Research Council.

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THE RARE OCCURRENCE OF MITOSIS WITHOUT SPINDLE APPARATUS ("COLCHICINE MITOSIS") PRODUCING ENDOPOLYPLOIDY IN EMBRYOS OF THE AXOLOTL*

BY G. FANKHAUSER AND R. R. HUMPHREY

DEPARTMENT OF BIOLOGY, PRINCETON UNIVERSITY, AND DEPARTMENT OF ANATOMY, SCHOOL OF MEDICINE, UNIVERSITY OF BUFFALO

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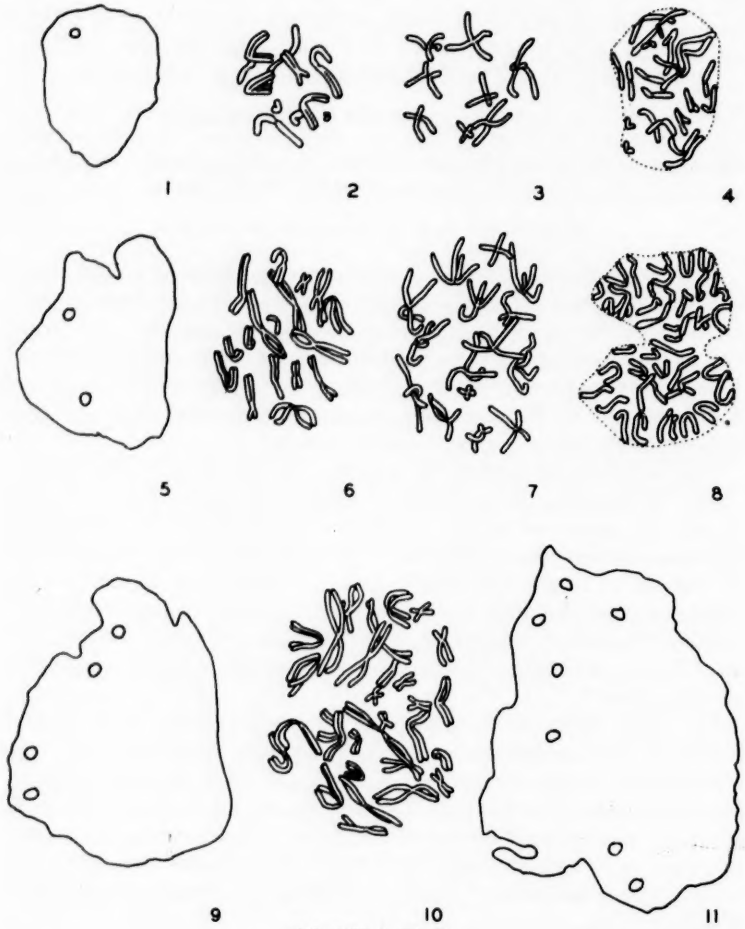
Endopolyploidy in Animals.—The occurrence of polyploid cells or tissues in otherwise diploid organisms, a condition known as polysomaty or endopolyploidy,† has received increasing attention during the past years. The fact that deviations from the normal chromosome number normally occur in the soma of many plants and animals has become so well established that it has led to the perhaps precocious generalization that uniformly and completely diploid organisms may not exist.

Endopolyploidy appears to be a regular phenomenon in differentiated tissues of flowering plants (Geitler,[5] Berger,[6] Huskins[7]) and in various organs and tissues of insects (Geitler[5]). In mammals, endopolyploidy has been described as a typical condition in the liver of the rat and mouse (D'Ancona,[8, 9] Beams and King,[10] Biesele,[11] Wilson and Leduc[12, 13]), in megakaryocytes and other types of blood cells (Ries,[14] Potter and Ward,[15] LaCour,[16] Schwarz[17]) and in cells from the spleen and kidney in tissue culture (Fell and Hughes[18]). It is also found frequently in tumor cells (Levan and Hauschka[19]).

As regards other classes of vertebrates, the information is at present limited to the amphibians, where endopolyploidy does not seem to be typical for any particular organ or tissue and, as a whole, may be a relatively rare phenomenon. It has been observed occasionally in liver cells of old salamanders (Mancini,[20] Barigozzi[21]), in kidney tubules of frogs (Dawson,[22] Schreiber and Melucci[23]) and, more frequently, in epidermal cells of the larval tailfin (Fankhauser,[24] Costello and Henley,[25] Henley and Costello,[26] Blair[27]).

Mechanisms Producing Endopolyploidy.—Endopolyploidy may be produced by any one of several types of deviations from the normal mitotic processes: (1) failure of cell division following normal mitosis, leading to the formation of binucleate cells which, at the following mitosis, may form a single mitotic figure with the tetraploid number of chromosomes; (2) more or less complete inhibition of the spindle apparatus, similar to that produced experimentally by colchicine and other chemicals, preventing normal anaphase separation of the chromosomes and producing restitution nuclei with the double number of chromosomes ("c-mitosis"); (3) "endomitosis" (in the wider sense), reduplication of the chromosomes within



FIGURES 1 TO 11

Stages in three successive "c-mitotic" cycles from epidermal cells of two axolotl embryos from the same spawning which presumably began their development with the haploid chromosome number (14). The two tailtip preparations do not contain any normal mitotic figures. No trace of spindle or asters visible. The nuclear membrane is present in some, but absent in other figures. Figs. 1, 2, 5, and 8 to 11 are from embryo 481–4, figures 3, 4, 6, and 7 from embryo 481–3. For the sake of clarity some of the chromosomes have been omitted from all drawings.

All figures are camera lucida drawings made at a magnification of 2525× which was reduced to approximately 1100× in reproduction. 1. Haploid interphase nucleus with a single nucleolus. 2. Haploid "metaphase" with 14 chromosomes plus one fragment; 10 chromosomes are shown in the drawing. Early stage of reduplication

the intact nuclear membrane. Of these mechanisms, endomitosis seems to be by far the most important for somatic polyploidization. Two main types of endomitosis may be distinguished, depending on whether the reduplication of the chromosomes takes place at a visible or submicroscopic level. In plants, endomitosis is almost always invisible, i.e., reduplication takes place while the chromonemata are in a partially uncoiled interphase condition. In such cases, evidence for endomitotic doubling is furnished by the increase in size of the nuclei, by the presence of the tetraploid or a higher chromosome number when the cells divide again by normal mitosis later on, and by the frequently paired arrangement of the sister chromosomes in the metaphase plate of such mitoses. An exception to this rule is provided by the tapetal cells of Spinacia in which the chromosomes become condensed and visible within the nucleus and can be followed through all stages of endomitosis (Witkus[28]).

Polyploidization by repeated reduplication of the chromonemata without visible change in the nucleus also occurs in insects; namely, in the cells of the hindgut of Culex larvae, which grow to large size and eventually demonstrate their polyploid condition when they divide again during metamorphosis (Berger[29]). However, endomitosis in insects is more often of the visible type first described and named by Geitler[30, 31] in various tissues of Hemiptera (Gerris, Lygaeus). The chromosomes are condensed but do not reach the degree of contraction typical for the metaphase of normal mitosis. Particularly characteristic are the endo-anaphases in which the daughter chromatids separate completely but remain more or less close together and parallel to one another. Stages suggestive of endomitosis have also been described in muscle cells of some nematodes (Vejdowski[32]), and in giant ganglion cells of pulmonate snails (Heitz[33]).

An abbreviated endomitotic cycle was discovered by Painter and Reindorp[34] in the nurse cells of the ovary of Drosophila which become highly polyploid but do not show complete separation of the chromatids. The

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of chromosomes. 3. Later stages of "metaphase" with 14 pairs of chromatids held together at point of spindle attachment ("diplo-chromosomes"); 8 chromosomes only are shown. 4. "Anaphase"; the daughter chromatids have separated completely but are still close together. Nuclear membrane present; 28 single chromosomes could be counted, of which 22 are shown. 5. Diploid interphase nucleus with two nucleoli. 6. "Metaphase" of second cycle; 28 double chromosomes are present of which 17 are shown. Nuclear membrane not clear, but confinement of chromosomes within oval space suggests its presence. 7. Later "metaphase" of second cycle with 28 typical diplo-chromosomes of which 15 are shown. 8. "Anaphase" of second cycle; as in figure 4, the daughter chromatids have separated completely but still show a paired arrangement. Only 42 of the 56 single chromosomes present are shown. 9. Tetraploid interphase nucleus with four nucleoli. 10. "Metaphase" of the third cycle; 30 of the 54-plus diplo-chromosomes counted are shown. 11. Octoploid interphase nucleus with seven nucleoli.

chromatids produced in previous division cycles remain close together in bundles and undergo mitotic changes reaching a very late prophase stage. There is no metaphase plate nor anaphase movement of the chromatids. In later division cycles the bundles of chromatids are looser than in the earlier ones but still do not break apart completely. This condition is clearly intermediate between cases of typical endopolyploidy, with separate chromosomes in polyploid number, and the formation of "multiple-strand" or "polytene" chromosomes in which the reduplicated chromonemata remain uncoiled and intimately associated in bundles to form giant chromosomes, as in the salivary glands and other tissues of larvae of Diptera.

The information on the mechanisms producing endopolyploidy in vertebrates is less complete. In the liver and other organs of mammals, the production of binucleate cells and the formation of a single, tetraploid mitotic figure during the following mitosis have been observed in sections of rat liver (Reams and King[10]) and in living cells from mouse spleen and kidney in tissue culture (Fell and Hughes[18]). However, in the mouse liver, polyploid cells may be produced also by an endomitosis or c-mitosis-like process, as is shown by the occurrence of prophase-like nuclei with short, compact chromosomes and of single, isolated nuclei that are in telophase (Wilson and Leduc[12]). These authors suggest that, during the postnatal development of the mouse liver, the mitotic process is suppressed in progressively earlier stages. Failure of cytoplasmic division occurs first; in older animals, mitosis is inhibited at metaphase by suppression of the spindle; still later, endomitosis may occur without breakdown of the nuclear membrane.‡

In mouse tumors, cells with polyploid chromosome complexes occur frequently, as well as cells the chromosomes of which are twice or four times the normal volume. There is evidence from the presence of an increased number of nucleoli that these enlarged chromosomes possess multiple chromonemata with multiple nucleolar organizers. Both polyploidy and polyteny often occur together and thus are probably produced by similar endomitotic processes. Certain stages of endomitosis corresponding to metaphase and anaphase have been described (Biesele, Poyner and Painter[35]). Evidence for the existence of double or multiple-stranded chromosomes which would imply the occurrence of repeated intranuclear reduplication of chromonemata has also been found in tumors of several other species of mammals and of the goldfish (Biesele[11, 36–39]).

As regards the origin of endopolyploidy in amphibians, evidence is available for the existence of at least two mechanisms. Partial failure of the spindle in some cells is indicated by the occurrence of more or less broadly connected telophase nuclei in the epidermis of the tailfin of larvae, both untreated, and, more frequently, after exposure to cold (Barber and

Callan[40], Costello and Henley[25]). On the other hand, "arrested" or "relapsed" metaphases have been described with varying degrees of separation of the chromatids which may enter telophase directly (Barber and Callan[40], Elair[27]).

Endopolyploidy in Axolotl Embryos and Its Origin.—In the course of the last ten years we have examined tailtip preparations of several thousand axolotl larvae (Ambystoma or Siredon mexicanum), obtained from various types of matings, to determine the number of chromosomes present. In addition to uniformly diploid or heteroploid tailtips, a number of mosaics were found, most frequently of the haploid/diploid type. As a rule, these mosaics were chromosomal "chimeras," with the two chromosome numbers occupying separate large areas, e.g., the right and left sides of the body, respectively. Such mosaics must be produced by accidents during fertilization or very early cleavage (Humphrey and Fankhauser[41]). Not infrequently, diploid tailtips with scattered heteroploid cells were also seen but did not furnish definite clues concerning the origin of the abnormal chromosome complexes.

In 1951, three embryos were discovered in the same spawning whose tailtips showed a different kind of mosaicism. The nuclei varied greatly in size and often were irregular in shape without showing a definite pattern of distribution. Mitotic figures were abundant. However, a closer inspection showed that they were all of an unusual type, involving chromosome reduplication without nuclear or cell division. In none of these cells could a spindle or asters be seen. In some cases, the chromosomes were clearly confined within a nuclear membrane. In others, the nuclear membrane seemed to be absent and the chromosomes scattered in the cytoplasm. While a complete cycle of reduplication thus may have taken place in some cells within the intact nuclear membrane, this membrane disappeared in other cells, at least during "metaphase," so that the term "endomitosis" cannot be applied generally. Actually, in its details, the abnormal mitotic cycle resembles more closely that described in cells treated with colchicine and other spindle inhibitors and is thus designated here as a "*c*-type mitosis." Whether or not the nuclear membrane actually disappears is of secondary importance; it may well be that both true endomitosis and c-mitosis occur, representing different degrees of mitotic inhibition.

The tailfin of one embryo which must have been diploid originally contained large areas of polyploid cells with up to eight nucleoli. The other two embryos were originally haploid and still showed a predominance of haploid cells that were mixed with diploid, tetraploid and some octoploid ones. In the latter two cases, three successive cycles of chromosome doubling must have taken place in some of the cells to increase the chromosome number from the haploid to the octoploid.

Various stages of these cycles are represented in figures 1 to 11. Following a normal prophase with chromosomes in the haploid number (Fig. 2) the chromosomes are distributed irregularly and show a progressive separation of the free ends of the chromatids (Fig. 3). Because of a delay in the division of the centromeres, typical cross-shaped "diplo-chromosomes" are formed. Later on, the centromeres divide and the two chromatids separate completely; however, the grouping of homologous chromosomes in pairs and their frequently parallel arrangement clearly demonstrate their common origin (Fig. 4); the diploid number of single chromosomes (28) can be counted. The telophase transformation of the chromosomes could not be followed in detail since telophases were rare; however, all telophase nuclei that could be identified were relatively large and single.

Figures 5 to 8 represent stages in the second reduplication cycle, leading from the diploid to the tetraploid condition. Again, the cross-shaped diplo-chromosomes at "metaphase" and the paired arrangement of the sister chromosomes at "anaphase" are typical (Figs. 7 and 8). Large tetraploid nuclei with four nucleoli were quite numerous. Stages of the third cycle were limited to nuclei containing the tetraploid number of early diplo-chromosomes (Fig. 10). No anaphases with the octoploid number of single chromosomes were present; but several very large, presumably octoploid, nuclei with up to seven nucleoli were found.

In the tailtip that was initially diploid, evidence for two "c-mitotic" cycles leading to the production of octoploid nuclei was plentiful, but no higher degrees of polyploidy could be detected.

It should be emphasized that all three embryos were abnormal and nonviable. A preliminary study of serial sections made by Mr. A. Alan Humphries showed that polyploid cells are present in all organs and tissues throughout the body of the embryos, with stages of "c-mitosis" evident particularly in the epidermis. This same spawning, in addition to a large number of apparently normal diploids also contained a typical haploid whose tailtip showed no signs of endopolyploidy. The cause of the abnormal mitotic behavior of the two other haploids and of the one diploid remains unknown. Neither is it possible to say at what time during development the abnormal form of mitosis first appeared.

The same type of polyploidizing mitosis has since been found in two other tailtips, basically diploid, from two different spawnings. In one of these, various stages of normal mitosis are also present, indicating that the conditions leading to the c-mitotic type can develop in single cells, or in groups of cells, rather than in the embryo as a whole.

Discussion.—As has been mentioned before, the process of chromosome doubling found in axolotl embryos resembles a "colchicine-mitosis" more closely than the endomitosis described in Hemiptera. Not only does the nuclear membrane break down during "metaphase," in some of the cells

at least, but the degree of contraction of the chromosomes is more complete than in insects and seems to reach a normal metaphase condition. The delay in the division of the centromeres which leads to the formation of typical diplo-chromosomes with widely divergent chromosome arms closely resembles the effects produced by colchicine in mitoses of Allium as originally described by Levan[42] and Berger and Witkus.[43] The parallel arrangement of the sister chromosomes at "anaphase" which follows upon the delayed division of the centromeres, and the formation of single telophase nuclei, are also similar in both cases.

A comparison of our c-type mitosis with the modifications of mitosis induced by colchicine-treatment of amphibian cells shows some important differences. Bureau and Vilter[44] picture typical diplochromosomes in epithelial cells of axolotl larvae treated with colchicine. However, these mitoses seem to be permanently blocked in this stage, and the centromeres remain undivided. In epidermal cells of newt larvae, Barber and Callan[40] describe "star" and "ball" metaphases, with closely bunched chromosomes, as the commonest effects of colchicine. Typical diplo-chromosomes are not formed. The arrest in metaphase is followed either by a lapse of the chromosomes into a normal resting nucleus, accompanied by a division of the centromeres, or by degeneration. In the cornea of adult *Triturus viridescens*, colchicine also produces clumped and star metaphases which seem to revert to interphase without passing through anaphase (Peters[45]). During recovery from the colchicine treatment, typical diplo-chromosomes may appear, as well as pairs of separate chromatids. In larvae of Xenopus, mitoses are blocked in metaphase with clumped and highly contracted chromosomes (Lüscher[46]). The majority of these cells become pycnotic and are phagocytized. Only rarely is a polyploid nucleus reconstituted.

In view of the initially haploid constitution of two of the axolotl embryos showing endopolyploidy it is interesting to mention corresponding observations on somatic doubling of the chromosomes in haploid, parthenogenetic grasshopper embryos. While some of these are either entirely haploid or entirely diploid, others contain both haploid and diploid, and sometimes more highly polyploid cells. Chromosome doubling must take place in somatic cells at various stages during development, probably by a process of endomitosis. This mechanism is suggested by the occurrence of haploid cells in metaphase with widely split diplo-chromosomes, and of diploid cells in which the homologous chromosomes tend to lie near one another (King and Slifer,[47] Swann and Mickey[48]). To our knowledge, other stages of the endomitotic cycle have not been described in these embryos.

As far as we know, the c-mitosis-like process leading to endopolyploidy discovered by us in axolotl embryos is the most completely documented case in vertebrates of a somatic reduplication cycle of the chromosomes in the absence of nuclear and cell division. The cytological advantages of

this material have made it possible to analyze the critical stages and to check the progress of repeated chromosome reduplication by accurate chromosome counts. Our findings support and complete the partial evidence given by other investigators that a process of chromosome doubling, either with or without breakdown of the nuclear membrane but always in the complete absence of any spindle, may play an important role in the production of polyploid cells in normal and tumor tissues of mammals. Once more it should be stressed, however, that, in axolotl embryos, this c-mitosis-like process is a rare and abnormal phenomenon and not a process connected with the constant appearance of endopolyploidy in certain organs or tissues, a condition which has not as yet been demonstrated to occur in amphibians. As a rule, the chromosome complement with which the fertilized amphibian egg begins its development, whether diploid or heteroploid, seems to be reproduced with a high degree of accuracy during the many cell divisions that take place during development and growth.

Summary.—Endopolyploidy, the occurrence of polyploid cells or tissues in otherwise diploid animals, is a regular phenomenon in many insects and in the liver, and possibly other organs, of mammals. It may originate from (1) intranuclear reduplication of the chromosomes ("endomitosis"), (2) failure of the spindle apparatus ("colchicine-" or "c-mitosis") or (3) failure of cell division and formation of binucleate cells with subsequent establishment of a single mitotic figure at the next division.

In the course of the study of tailtip preparations of several thousand axolotl embryos, three abnormal embryos were discovered that showed a high degree of endopolyploidy; two of these were originally haploid but also contained diploid, tetraploid and octoploid cells; the third embryo was originally diploid and showed an admixture of tetraploid and octoploid cells. All mitotic figures in the three embryos were of an abnormal c-mitosis-like type, without spindle apparatus, with delay in the division of the centromeres leading to the formation of typical diplo- chromosomes, and with reconstitution of single nuclei containing the double number of single chromosomes. Stages in three successive c-mitotic cycles raising the chromosome number from the haploid (14) to the octoploid were found. This rare but spontaneously occurring type of mitosis without spindle apparatus which could be analyzed in detail because of the cytological advantages of amphibian cells, may play a role in somatic chromosome doubling in mammals, along with endomitosis and the formation of binucleate cells.

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† The relation between these two terms has not been completely clarified (cf. Mickey,[1] Lorz,[2] Geitler[3]). "Polysomaty" refers to the regular occurrence of both diploid and polyploid cells in the same tissue and has been limited in its use almost entirely to plants. "Endopolyploidy" is used for all cases of somatic doubling of the chromosome number in animals which lead to the presence of polyploid cells in diploid individuals, or of diploid and polyploid cells in haploids, whether the particular tissue or organ be entirely or only partly polyploid (cf. White[4]). The resemblance to the term "Endomitotische Polyploidisierung" coined by Geitler[3] might imply that endopolyploidy is always produced by intranuclear chromosome doubling ("endomitosis"). However, it seems logical to use the term in a wider sense to include those cases of partial somatic polyploidy in which the mechanism involved either is as yet unknown or consists of total or partial inhibition of the spindle apparatus following the breakdown of the nuclear membrane ("colchicine-" or "c-mitosis"). It is probable that forms of polyploidizing mitosis transitional between endomitosis and c-mitosis occur, or that both may occasionally be found in the same tissue. If, in addition to cells with multiples of the basic chromosome number, aneuploid cells, with unbalanced chromosome complements are present, the term "mixoploidy" proposed by Mickey[1] may be applied.

‡ Since this article went to press, stages of endomitosis have been described in embryonic chick cells grown in vitro at 42 to 43°C. (E. F. Stilwell, *Anat. Rec.*, **114,** 9–18), and c-mitotic metaphases, probably induced by hypotonicity of the washing fluid, have been found in cultures of skin from a human embryo (T. C. Hsu, *Jour. Hered.*, **43,** 167–172 (1952)).

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